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GRAVITY AND MATRIX BRANES

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ABSTRACT. This is a article that discusses how a gravity model previously proposed gives a M(atrix) model for branes. I also discuss the issue of Stein manifolds in physics.

1. The Model We Begin With

The model we begin with is, without ghost terms, symmetry braking terms and no effective action terms;

$$L_{total} = |\not{D}\ \theta|^2 + \bar{\psi}(i\not{D}\ - m)\psi + \frac{|F+R|^2}{4}$$

where the pure gravity lagrangean is

$$L_{gravity} = |\not{D}\ \theta|^2.$$

θ is the vielbein, $D = d+\omega+A$ is the covariant connection, ω is the Cartan Riemannian connection after Wick rotation, A is the gauge potential, ψ and $\bar{\psi}$ are the Dirac spinors, and $R = d\omega + \omega \wedge \omega$ is the Riemann curvature tensor and $F = dA + A \wedge A$ is the gauge field strength. The various coupling constants are set to 1 for simplicity. The theory we look at lives in dimension 4, which we now complexify by adding a small $i\epsilon$, where i is the imaginary unit and ϵ is a small real constant to the transverse direction of this $D = 4$ real-analytic space-time. Thus we are now in complex dimension $D^{\mathbb{C}} = 4$, which is real dimension $D = 8$. Adding a flow parameter on this manifold would give $D^{\mathbb{C}} = 5$ or $D = 10$. It is understood that this flow is complex holomorphic.

2. The Model We Get

We understand that θ then becomes a complex vielbein that corresponds to a hermitean metric structure on $X^8 = (X^4)^{\mathbb{C}}$ our complexified manifold X^4. This manifold happens to be Stein. Now, the complex vielbeins $\mathbb{X}$ are given by doing a unitary diagonalization of the hermitean metric $h = h_{\mu\bar{\nu}}$. So we set

$$h = \mathbb{E}^{\dagger}\delta\mathbb{E}$$

where $\delta > 0$ is a real diagonal matrix and $\mathbb{E}$ is unitary. This δ is real because h is hermitian, and it is positive because we work with a well-defined Euclidean metric after Wick rotation. And, of course, the complex vielbein $\mathbb{X} = \sqrt{\delta}\mathbb{E}$ can be chosen so that it is unitary if $\delta := 1$, so that $\mathbb{X} \in U(N)$. In the general case $\mathbb{X} \in \sqrt{\delta}U(D)$. Then

$$L_{gravity} = |\not{D}\ \theta|^2$$

becomes after this complexification

$$L_{gravity} = |\not{D}\ \mathbb{X}|^2.$$

Let us use the Weizenbrock-Bochner identity

$$\not{D}\ ^2 = \Box + \frac{R}{4} + \not{F}\ .$$

Assume that the moduli of the topological type X are very coarse, like most higher dimensional complex holomorphic manifolds. Then a brane would be the sum of QFT's of all the holomorphic structures. We now get for the first few terms of this matrix model with $\delta := 1$, where $\mathbb{X}$ is now the inverse $\mathbb{X} = (\sqrt{\delta}\mathbb{E})^{-1}$ and h is normalized

$$L_{matrix} \sim tr(\mathbb{X}^\dagger \Box \mathbb{X} + \mathbb{X}^\dagger \mathbb{X}\frac{Ricci}{4} + \mathbb{X}^\dagger \not{F}\ \mathbb{X} + m_{eff}^2 \mathbb{X}^\dagger \mathbb{X} + \Lambda + \mathbb{X}^\dagger V(\mathbb{X})\mathbb{X} + ...)$$

where we added a cosmological term Λ, a effective mass m_{eff} and as well as a matrix potential $V(\mathbb{X})$ that is quadratic up to the first term, yielding a quartic term as in Polchinski[10]'s description of matrix theory and D-branes, see page 211 in volume II. For a non-normalized inverse metric $G = h^{-1}$ this is

$$L_{matrix} \sim tr(\mathbb{X}^\dagger G \Box \mathbb{X} + \mathbb{X}^\dagger G \mathbb{X}\frac{Ricci}{4} + \mathbb{X}^\dagger G \not{F}\ \mathbb{X} + m_{eff}^2 \mathbb{X}^\dagger G \mathbb{X} + \Lambda + \mathbb{X}^\dagger G V(\mathbb{X})\mathbb{X} + ...)$$

We can further generalize to non-hermitean metrics and obtain for example for the integration measure in the action

$$\sqrt{h} d^D x \sim det(G+B)^{1/2} d^D x \sim det(G + B + 2\pi\alpha' F)^{1/2} d^D x$$

where G is now the metric and not the inverse metric. The last term is the root of the Chern class polynomial and is also the Dirac-Born-Infeld action, and $\alpha' = 1/T$ is the slope. The nature of the metric strongly affects the stability and out-channel dynamics and decay.

3. Properties of this Matrix Model

We note that when we have the pure gravity Lagrangean

$$L_{gravity} = |\not{D}\ \theta|^2$$

the θ^a's can be made to describe brane fields from exterior monomials

$$\theta^{a_1} \wedge \cdots \wedge \theta^{a_n}$$

Now, these fields are in the Matrix description

$$\mathbb{X}^{a_1} \wedge \cdots \wedge \mathbb{X}^{a_n}$$

which describes a $n-1$-brane.

4. Critical Dimension

The action

$$L_{gravity} = |\not{D}\ \theta|^2$$

above has critical dimension $D = 4$. The Matrix model

$$L_{gravity} = |\not{D}\ \mathbb{X}|^2.$$

has holomorphic critical dimension $D^{\mathbb{C}} = 4$. So the matrix model is a complex holomorphic model, which justifes the Wick-rotation procedure.

5. The Stein Manifold Property and The Maldacena/Holography Property

In a talk at the Mittag-Leffler Institute I stated that the holomorphic Matrix model above should be chosen to be Stein, that is

(1) $X^8 = (X^4)^{\mathbb{C}}$ is a complex holomorphic manifold.
(2) X^8 is holomorphically convex.
(3) X^8 has global point-separating holomorphisms.
(4) X^8 has global holomorphisms as local charts or patches.

This gives the property that

(1) $X^8 = (X^4)^{\mathbb{C}}$ is a complex manifold which has the physics of is a real $D = 4$ manifold. For example, if you let the bulk have trivial topology you will understand that any topological theory in effective dimension 8 or dimension 10 displays holography to dimension $D = 4$, which I call the Maldacena property.
(2) This is a obvious dimensional reduction.
(3) Wick-rotations and analytic continuations of amplitudes are well-defined by theorems of several complex variables.

and you have to notice that this is the exact opposite of the compact situation that is chosen by many physicists and mathematicians. It's more or less at the other side of complex manifold behaviour. Mathematically this dimensional reduction has been known for long as the Oka principle.

6. Holomorphic Factorization

The holomorphic model above satisfies the well-known holomorphic factorization principle in string theory, albeit now in the brane setting. This is known to me as the D'Hoker-Phong theorem, readily accessible in the literature, see Witten et al [7]. That is, any brane correlator W can be seen as a holomorphic factorization

$$W = C\bar{C}$$

For example, the partion function is

$$Z_{total,D=10} = Z_4\bar{Z}_4.$$

I argued in my book that holomorphic factorization is very important in gravity and string theory, see Torbrand Dhrif[3]. This also is related to something I call algebraic duality and gauge theory-string theory correspondences. For example to 1-loop or genus $g = 1$ this is

$$Z_{total,D=10} = 16\Pi_n \frac{(1+q^n)^8}{(1-q^n)^8} = (4\Pi\frac{(1+q^n)^4}{(1-q^n)^4})^2 = Z_4\bar{Z}_4.$$

where

$$Z_4 = 4\Pi_n \frac{(1+q^n)^4}{(1-q^n)^4}$$

where q is real. Analysis of Z_4 reveals that it is related to to SUSY gauge theory in $D = 4$. The square $Z^4 \times \bar{Z}^4$ comes from the probability interpretation of quantum theory-that is, taking squares of wave-functions gives probability densities-and is a very nice motivation for these dualities. In my book I called these two different situations the half-density and full-density situations or theories, see Torbrand Dhrif[3]. In string scattering theory it has been known for long that one describes physics in terms of full-densities, where as gauge scattering theory requires the norm squared of S- matrix elements, that is half-densities.

7. The Moduli of These Brane Theories

Actually, in the complex geometry setting to brane theory, pending on the nature of the situation, the moduli can be very big or very scarce. A analytic hypersurface for example can have very large moduli, where as a compact Riemann surface has finite dimensional moduli.

8. The Nature of Einstein Branes

In my articles Torbrand Dhrif[5] and Torbrand Dhrif[6] I state some of the properties of Einstein-Branes, which I call E-branes. For Einstein Branes the partition functions reduce to a single topology type because of a hidden diffeomorphism (biholomorphism) symmetry for the actions of these real analytic $D = 4$ branes we begin with, or ($D = 10$ or $D_{eff} = 8$ holomorphic manifolds). Since in the Stein situation these $D = 10$ models will be very loose this is exactly the right tool to compute amplitudes, at least when we have a nice deformation or homotopy of these manifolds. So basically we are always looking at 3-branes up to diffeomorphism, which we then complexify- that is our trick.

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