GRAVITY IN D=4, NAIVE RENORMALIZABILITY, CFT'S AND DUALITY

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Abstract. This is a sequel to the article 'A More or Less Well-Behaved Quantum Gravity Lagrangean in Dimension 4?' in Advanced Studies in Theoretical Physics, Torbrand Dhrif[6]. I propose a Quantum Gravity Lagrangean, and I first work with a simplified linear version of the non-linear theory I propose. I hope that Feynman calculus behaves well for these theories. These two actions are naively renormalizable(a concept defined in the article, more or less equivalent to the notion of a CFT) in dimension 4 and have critical dimension 4, although the non-linear theory is a CFT in every dimension and so every dimension is critical for it. The non-linear theory is equivalent to Einstein gravity in the interaction picture up to a redefinition of the coupling constant. It is also expected to be close to some subsectors of IIA string theory, and relates to M(atrix) Theory. As applications of these theories I show two theorems; One that is a Maldacena type theorem and another which is gauge-gravity duality, both with $D = 4$ space-time but with far more general geometry. I also go through gauge-gravity dualities in arbitrary D but in other settings. This is still a working paper and thus not finished.

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1. Gravity Quantized in Dimension 4

We assume standard notation such as that $\omega = \omega_a^\alpha \theta^a T_\alpha$ is the ON-connection, T_α are generators of the Lie algebra $\mathfrak{so}(1, D-1)$ in the defining representation, θ^a are the vielbeins, $\wedge$ is the wedge product, g is the metric or a dimensionless gravitational coupling constant as apparent from the situation, R is the Riemann curvature or the the Ricci scalar curvature pending on situation, *Ricci* is always the Ricci scalar curvature, A is the gauge potential, F is the gauge field strength, Γ^a are generators of the Dirac or Clifford algebra, d is the exterior derivative, D is either the Riemannian (pseudo-Riemannian) covariant ON-connection with color, so $D = d + \omega + A$, or the dimension of space-time. This real space-time M is assumed to be complexified to $M^{\mathbb{C}}$, so that we have added very small negligible imaginary directions and made it holomorphic. D is then the dimension prior to complexification. The δ^{ab} and η^{ab} are the Kronecker deltas in various signatures. $\Box$ is the connection D'Alembertian or Laplacian (pending on signature) defined as in the appendix. $*$ is either the dual or the Hodge star. When I write (physical) dimension, such as in dimensionless or dimensionfull, I usually mean it in the sense of physical length or mass dimension.

Notice the following Bochner type formula for compatible spinor connections on spinor fields

$$\not{D}^{\,2} = \Box + \not{R} + \not{F} = \Box + \frac{Ricci}{4} + \frac{F_{ab}[\Gamma^a, \Gamma^b]}{4},$$

where *Ricci* is the curvature scalar, see Lawson and Michelsohn[12], page 160, page 164 and page 398, or Jost[16], page 145. Here

$$R = \frac{R_{ab}}{2}\theta^a\theta^b = \frac{{R_{ab}}^\alpha}{2} T_\alpha \theta^a\theta^b = \frac{{R_{ab}}^\alpha}{2} T_\alpha \theta^a \wedge \theta^b$$

is the Riemann curvature tensor, a Lie algebra valued 2-form, with T_α a basis of the defining representation of the Lie algebra $\mathfrak{so}(1, D-1)$. F is the gauge field strength for $U(N)$ or similar gauge group. This formula for the square of the Dirac operator $\not{D}$ will be the inspiration to much of the ideas in this article. We want to find something similar on vielbeins, and also relate this to vielbeins. We use the Dirac or Clifford algebra representation

$$\Gamma^a = \theta^a + \theta^{*a}$$

so that

$$[\Gamma^a, \Gamma^b]_+ = \{\Gamma^a, \Gamma^b\} = 2\eta^{ab}.$$

We now work with Euclidean signature, so $\eta^{ab} := \delta^{ab}$. This does not affect the discussion, albeit Lorentz signature has to be used in the physical case. Here $\theta^{*a} := \bar{\theta}^{*a}$ and θ^a generate a finite dimensional Fock-algebra via

$$\delta^{ab} = [\theta^{*a}, \theta^b]_+ = \{\theta^{*a}, \theta^b\} = \theta^{*a}\theta^b + \theta^b\theta^{*a}$$

and

$$0 = [\theta^a, \theta^b]_+ = \{\theta^a, \theta^b\} = \theta^a\theta^b + \theta^b\theta^a.$$

Also

$$0 = [\theta^{*a}, \theta^{*b}]_+ = \{\theta^{*a}, \theta^{*b}\} = \theta^{*a}\theta^{*b} + \theta^{*b}\theta^{*a}.$$

[1] For a good account of Clifford algebras and related matters, see Jost[16], page 55-75. Jost[16] defines on page 68 spinor space S over a point p in space-time $M^{\mathbb{C}}$ as $S = \wedge T_p^{*+} M^{\mathbb{C}}$, the fiber of the holomorphic exterior cotangent bundle over that point [2]. This is very closely related to the vielbeins on space-time since the holomorphic vielbeins with complex coefficients over that point generate this exterior algebra. The $\theta = [\theta^a] = [\theta^a_\mu dx^\mu]$ is a smooth enough with x^μ a real coordinate(and not holomorphic intially) ON basis on the pertaining manifold. This is also a vector-valued one-form, the components are the vielbeins. The θ^{*a} are then the dual vielbeins as above. For the colored vielbeins used later see the appropriate appendix. This is in the notation and idea of E. Cartan, who wrote the metric g in terms of the veilbeins θ^a_μ as $g_{\mu\nu} = \theta^a_\mu \delta_{ab} \theta^b_\nu$. Equivalently $g(\theta^a, \theta^b) = \delta^{ab}$ or

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = g(\hat{e}_a, \hat{e}_b) \theta^a \otimes \theta^b = \delta_{ab} \theta^a \otimes \theta^b$$

where $\hat{e}_a$ is the ON basis in vector form. From now on we let latin letters $a, b, c, \cdots$ denote expressions in the ON-frame and greek letters $\mu, \nu, \sigma, \cdots$ denote expressions in the coordinate frame. We set the vielbein Dirac operator as

$$\not{D} = \Gamma^\mu(\partial_\mu + \omega_\mu + A_\mu) = \Gamma^\mu(\partial_\mu + \omega^\alpha_\mu T_\alpha + A^\beta_\mu t_\beta) = \Gamma^a D_a$$

where ∂_μ is partial differentiation with respect to the real coordinate x^μ, ω_μ the ON-connection, $A_\mu := 0$ is the gauge potential and t_β are the generators of the Lie algebra $\mathfrak{u}(N)$ in the defining representation, with anti-Hermitean conventions[3]. See Green, Schwarz and Witten [19], page 387, volume II, for this Dirac operator representation [4]. The gauge potential is set to be zero because it does not affect the discussion below to any greater extent. We also set

$$\Gamma^a = \theta^a_\mu \Gamma^\mu.$$

Let us define the pure non-linear gravity action and Lagrangean as

$$S_{Gravity} = \int L_{Gravity} \sqrt{g} d^D x = \int |\not{D}\ \theta|^2 \sqrt{g} d^D x.$$

We thus conclude the following (see a side condition below) [5]

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[1]You could see this as $\theta^a := \theta^a \wedge$ the exterior product and $\theta^{*a} := i^a$ the interior product on a exterior algebra generated by the vielbeins θ^a over smooth functions.

[2]I set $V := T_p^* M^{\mathbb{C}}$ as the fiber of a real cotangent bundle over $M^{\mathbb{C}}$ as a real manifold, and $W := T_p^{*+} M^{\mathbb{C}}$ defined over the complex numbers, with his notation. Thus $V \otimes \mathbb{C} = W \oplus \bar{W}$.

[3]This gauge group is for real usually $U(1) \times SU(2) \times SU(3) \times \cdots$, and here we assume that the graviton does not carry such charges by seeting $A_\mu := 0$. That means no electromagnetic, (electro)weak or chromodynamic charge, so we can say $N = 0$, N the number of colors. It is natural to believe that there exist such charged exitations of the graviton however. Here I use the standard physics notation $\times$, which should be the tensor product $\otimes$.

[4]The Hodge representation of this Dirac operator on vector valued differential forms is $\not{D} = D - D^* = \Gamma^\mu D_\mu$, where D^* is the Hodge dual of D. Yes, the minus sign is correct.

[5]The inner product is defined in Hodge notation as $< \omega, \eta >= *(\omega^\dagger \wedge *\eta)$. ω and η are vector valued differential 1-forms. $\dagger$ is the Hermitean adjoint.

$$|\not{D}\ \theta|^2 = \theta^* \not{D}\ ^2\theta = \theta^*(\Box + \not{R}\ )\theta = \theta^*\Box\theta + R = Ricci$$

with R the Ricci scalar [6]. Here θ is the graviton or vielbein and the Ricci term arises similarly to Lawson and Michelson [12], page 156, Jost[16], page 137 and page 146. *Also you can check the 1:st appendix at the end of this article where I do the calculus for these expressions in the Cartan formalism that I use.* Notice that this leads to the Hilbert-Einstein action [7]

$$S_{Einstein} = \int_M L_{Einstein}\sqrt{g}d^Dx = \int_M Ricci\sqrt{g}d^Dx$$

when we assume the side condition

$$\theta^*\Box\theta = 0,$$

which basically states that gravitons are mass-less and propagate with the speed of light in a appopriate flat background with no external fields. So with our side condition,

$$L_{Gravity} := |\not{D}\ \theta|^2 = R = Ricci = L_{Einstein}$$

or in the interaction picture,

$$L_{Gravity,interaction} = R = Ricci = L_{Einstein}.$$

Remember that we exponentiate the interaction Lagrangean or Hamiltonian, pending on a sign, to obtain the matrix elements in QFT- say in the operator formalism- so we are not wrong, we are completely right! [8]

Let us treat the vielbein θ as a vector valued 1-form independent from the metric in this first analysis of the pure gravity action. I call this the simplified linear setting or theory. We will look at the full non-linear situation later in this article. The non-linear theory is very difficult to handle, and that is the reason for this first simplification. See section 5 and later. We now set the pure gravity simplified linear theory Lagrangean as

$$L_{Gravity,simplified} := |\not{D}\ \theta|^2_h$$

Here h is the background non-dynamical metric and the norm $|\cdot|$ and it's associated inner product $< \cdot, \cdot >$ only depend on the background metric h. The vielbein Dirac operator in this simplified theory depends on the background metric and does include connections. This background metric

[6]We sometimes suppress a minus-sign or a imaginary unit i in the following.

[7]Here we have supressed a term $16\pi G$, G the gravitational constant. We will in fact reformulate the gravity coupling constant as associated to the ON-connection instead, multiplying the ON-connection, similar to the standard formalism of gauge theory. The usual gravity coupling constant has physical dimension, unlike the coupling constants of the standard model. This screws up the dimension counting if you do not redefine it in this way, at least if you normalize the vielbeins. See remark 5.16 for thoughts about this. Also, if you do not normalize the vielbeins at all, and thus do not subscribe to most of the big assumptions of this article, the physical dimension calculus still goes through and gives Maldacena and Gauge-Gravity duality. See remark 5.12 for not assuming normalization of vielbeins, and remark 5.7 for Maldacena and Gauge-Gravity calculus.

[8]You could also see this as employing a artificial counter-term. These two theories are equivalent modulo this term.

h should not be confused with the full dynamical metric g. This simplified linear Lagrangean has physical mass dimension 4(Compare to a YM-term if you wish, say in QED, it has the same dimension and looks very similar.), thus naively renormalizable in dimension 4. This concept of naive renormalizability is defined in section 5. Let's prove this in a more obvious manner: Our volume measure is $\sqrt{g}d^D x$, and since we use the background metric only this becomes $\sqrt{h}d^D x$. Then our action functional becomes

$$S_{Gravity,simplified} = \int |\not{D}\ \theta|_h^2 \sqrt{h} d^D x.$$

We set $e^a := \theta^a / L^a$, where L^a is the length in direction a. This e^a has dimension 0 and is θ^a after a normalization to make it give probabilities which are normalized to a total probability of the number 1, look at the note below. Thus e^a_μ has mass dimension 1. Thus finally

$$(\theta^a_\mu)^*(\Box + \not{R}\ )(\theta^a_\nu)_{normalized} = (e^a_\mu)^*(\Box + \not{R}\ )(e^a_\nu)$$

has mass dimension 4 or length dimension -4. The space-time measure has length dimension D. So we have naive renormalizability by dimension counting in D=4, and the critical dimension is also D=4. See section 5 for the definitions of naive renormalizability and critical dimension.

Note the identity

$$\sqrt{g}d^D x = \theta^1 \wedge \theta^2 \cdots \wedge \theta^D = (T \times V)(\theta^1/T) \wedge (\theta^2 \wedge \cdots \wedge \theta^D)/V.$$

Here V is the volume of space and T is the length of time, and we set space-time to be tubular or foliated with spatial leaves of volume V. This becomes

$$\sqrt{g}d^D x = e^1 \wedge \cdots \wedge e^D \times V \times T.$$

For information on why the vielbeins can be interpreted as generating probability currents via exterior algebra see my book, Torbrand Dhrif [5], it's basically a statement of compatibility, or that the covariant gradient and covariant divergence of monomials generated by vielbeins vanishes. See the expressions

$$J_3 = *\theta^1/V = e^2 \wedge e^3 \wedge e^4, DJ_3 = 0, D^* J_3 = 0,$$

so we equate the current J to the above form. Evaluate the current on a cycle on when expressed in the orthonormal frame (θ^a). This is assumed that we pinpoint a fixed time where the graviton particle is measured with certainty, that is almost surely. Also note that the integral over some domain in space-time of the space-time volume form

$$J_4 = \frac{\theta^1 \wedge *\theta^1}{V \times T} = \frac{\theta^1 \wedge *\theta^1}{vol(M)} = e^1 \wedge e^2 \wedge e^3 \wedge e^4, DJ_4 = 0, D^* J_4 = 0$$

could have the interpretation of measuring the graviton particle in that time interval and part of space. Here $vol(M)$ denotes the total volume of space-time. This normalization physically means that the graviton will be detected somewhere in space-time. Of course this is a covariant statement in space-time. Most of the probability mass will be very near the wave front caused by the hyperbolic wave equation that the graviton satisfies, and that is a standard feature of quantum field theory. It might be good to first look at closed manifolds(compact manifolds) as space-time to make this calculus work. This is better than the 3-dimensional version in that it is completely covariant and has no problems of choosing 'space', and 'time' which is not necessarily easy. Also, if we carry the whole formalism through we need this to understand matrix elements of the S-matrix in probabilistic manner in position space by using the Hodge inner product.

If you set $\theta^0 = du = Du$, where u is some smooth enough function and d the exterior derivative, you end up very near to quaternionic geometry, see Hitchin et al. [11], page 9-30.

I now state the Lagrangean density for the standard model with gravity but not including symmetry braking terms, ghost terms, effective action terms, or mass terms, in the non-linear theory;

$$L_{total} = (|\not{D}\ \theta|^2 + \bar{\psi} i \not{D}\ \psi + \frac{1}{4}|R+F|^2)\sqrt{g}d^D x.$$

Here $\sqrt{g}d^D x = \epsilon_M = \theta^1 \wedge \cdots \wedge \theta^D$ is the volume form and measure, ψ and $\bar{\psi}$ are the Dirac spinors, $R = d\omega + \omega \wedge \omega$ is the Riemann curvature tensor with ω the ON-connection, $F = dA + A \wedge A$ is the gauge field strength or gauge curvature with A the gauge potential. Please do not confuse the two different Dirac operators in the Lagrangean above, one of them is the vielbein Dirac operator (with ON-connection) and the other is the spinor Dirac operator (with spinor connection), so they are two different mathematical objects. One can say that the module defines which kind of Dirac operator is used. Both of these Dirac operators can be assumed to come from a more general Dirac operator on a bigger space, hence the notation.

The linear simplified theory Lagrangean density for the standard model with gravity, without symmetry braking terms, ghost terms, effective action terms or mass terms on a background space-time with fixed metric h is;

$$L_{total,simplified} = (|\not{D}\ \theta|^2 + \bar{\psi} i \not{D}\ \psi + \frac{1}{4}|R+F|^2)_h \sqrt{h}d^D x.$$

I should point out that we have to work with L^2-spaces, that is well-defined Hilbert spaces, and also the mathematical theory of currents, which are generalizations of distributions to differential forms. In the most trivial applications and toy models one uses a compact smooth setting, but L^2-spaces of distribution valued forms are probably the first candidates for appropriate spaces for us to use in mathematical physics.

I summarize by the following;

- *These theories or actions are naively renormalizable by dimension counting and scale invariant in dimension 4. They have critical dimension 4. This is as good as the standard model in particle physics.*
- Note that the θ^a_μ follow bosonic commutation rules while the θ^a follow fermionic anticommutation rules. This is not a statement about spin.
- *Note that $L_{Einstein}$ is the interaction Hamiltonian or Lagrangean, pending on a sign, of our non-linear Lagrangean $L_{Gravity}$. So if you use the appropriate interaction picture it is exactly the same thing.*

- The calculus of the norm squared of matrix elements is evident. Feynman calculus for (unpolarized) cross-sections goes through, which is a great simplification, and produces metric terms which are much easier to handle in this theory.
- The vielbein and graviboson are the fundamental fields. The connection 'graviboson' has the property that the ON torsionless connection will solve the equation of motion obtained from the above pure gravity Lagrangean upon variation of the vielbein. This equation is [9] [10];

$$\not{D}^{\,2}\theta = 0.$$

Thus the equation of motion is that θ is harmonic[11]. *I will say that the gravitational fields are on-shell when they satisfy this equation of motion, this is standard physics nomenclature when a equation of motion is satisfied.* Actually there is a theorem on harmonic vector valued smooth differential forms on smooth compact manifolds that you should know. It states the following [12];

$$\not{D}^{\,2}\theta = -(D^*D + DD^*)\theta = 0$$

is equivalent to

$$D\theta = 0, D^*\theta = 0.$$

This means that the equation of motion for the pure gravity term in the Lagrangean is equivalent to requiring a compatible connection, basically the first Cartan structure equation with vanishing torsion;

$$d\theta + \omega \wedge \theta = D\theta = T = 0.$$

Thus the relation between the vielbein and the connection is as usual in Cartan's formalism or version of Einstein gravity. So, from the point of view of elementary physics this looks right. Please note that the ON-connection is not the same as the metric Levi-Cevita connection.
- I think we have relations to 8/10-dimensional superstring/supergravity models with certain versions of the Maldacena conjucture or theorem [13], See Torbrand Dhrif[7], as long as the bulk has trivial topology such

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[9] In correct signature this is a wave equation. It reduces to the classical linear wave equation $(\partial_t^2 - \Delta)\theta = 0$ asymptotically, with Δ the three dimensional flat Laplacian and ∂_t the partial derivative with respect to time, of course in flat Cartesian geometry. I leave it to the PDE aficionado to study this equation closer.

[10] I am using the simplified linear theory to derive this equation of motion.

[11] There are many kinds of notions of harmonicity in the literature, this is just one of them. It is apparent what we mean.

[12] The terms D^2 and D^{*2} are usually dropped as they vanish from the Lagrangean. This theorem holds for the full nonlinear $\not{D}^{\,2}$.

[13] The Maldacena conjecture and Gauge-Gravity duality to 1-loop turns out to be a study of trivial identities for determinants of the operator $\not{D}^{\,2}$. You will have to assume some linearization of the operator $\not{D}^{\,2}$ to do this. As I state in the appendix, there are clear supersymmetries in kinematical dimension 4, which one can use to prove such conjectures *on a single line*. Gauge-Gravity duality generalizes to much more general geometry in

as the forward hyperboloid of some of the simplest dS/AdS-spaces,

$$\rho^2 = t^2 - x_1^2 - x_2^2 - \cdots - x_n^2$$

all variables real, but with Wick rotations allowed in all coordinates, in the Green-Schwarz formalism after complexification of this 4 dimensional model. Particle physicists and quantum field theorists do work with a complexification by standard, e.g the Wick rotation procedure. Then the topology is namely the same, it's all in space-time, and this gives equivalent topological functors, like cohomological theories or physics expressed in cohomological theory such as BRST-theory, topological quantum field theories, anomaly theory or index theory. Remember, these theories could have pathologies or other peculiarities. In as far as string/sugra theory resemblences this is a good sign, because we think those models are at least somewhat nice. Yet our theory, notice well, lives in dimension 4.

- Notice that the string theory IIA Lagrangean (Use a Bochner type identity similar to the appendix) [14] is

$$L_{ST,IIA} = \int \mathcal{D}\bar{\mathbb{E}}^* \bar{\mathcal{D}}\mathbb{E} d\theta d\bar{\theta} + Ricci \sim |\not{D}\ E|^2 + \bar{\psi}(i\not{D}\ )\psi + \frac{1}{4}|R+F|^2,$$

with the superfield

$$\mathbb{E} = E^a_\mu \hat{e}^\mu \otimes \hat{e}_a + \bar{\psi}^\mu \theta \hat{e}_\mu + \bar{\theta}\psi^\mu \hat{e}_\mu + \frac{1}{2}(F+R)^{ab}\hat{e}_a \wedge \hat{e}_b \theta\bar{\theta}.$$

The operator $\mathcal{D}$ is defined as

$$\mathcal{D} = \theta e \frac{\partial}{\partial z} + \frac{\partial}{\partial \theta},$$

where

$$\frac{\partial}{\partial z}$$

is the Cauchy-Riemann operator or the free chiral Dirac operator on a Riemann surface(the string sheet), and e is the covariant holomorphic complex zweibein on this sheet. See Polchinski[17], page 105, volume II, or Witten et al [8], page 989, volume II, where we have to use E^a_μ for the vielbein instead to not abuse notation. The θ and $\bar{\theta}$ above are then anticommutative Grassmann variables. Here

$$\int (\cdots) d\theta d\bar{\theta}$$

is the Berezin or Grassmann integral. Here we put the Dirac fermion mass to $m = 0$ and the usual Fayet-Iliopolous term is instead set to the gauge and gravity field strength. That makes more

$D = 4$ if you use the gravity proposed in this article, and the Maldacena Theorem also generalizes in $D = 10$ with $D_{eff} = 8$ kinematical dimensions to more general geometry than usual. This is a clear sign that we are doing something right.

[14] We define the notation below, and there is a gauge potential A, a color term N and a graviboson ω supressed. Use the covariant Lagrangean on the right in general.

sense right here. Also, please do not confuse the two different Dirac operators above, one of them is the vielbein Dirac operator(with ON-connection) and the other is the spinor Dirac operator (with spinor connection). Both of these have gauge potential terms included and can be assumed to come from a more general Dirac operator on a bigger space, hence the notation. The really big gain is that our Lagrangean L_{total}, which is very close to the string theory IIA Lagrangean $L_{ST,IIA}$, that works in dimension D=10(with kinematical dimension 8 and kinematical codimension 2), works with critical dimension 4. That is really big news! It also makes it very probable that we have some very nice dualities to expect between these dimensions after complexification of the 4-dimensional model that we propose in this article.

- Complexification makes the graviton sector of this superbrane model a M(atrix)-brane Lagrangean on a Stein brane(manifold), see the article Torbrand Dhrif [14]. I tend to see branes as sums over moduli of QFT's. There are various mechanisms, see Torbrand Dhrif[15], that may make all that remains a QFT.

2. On the nature of the non-linear partial differential equations for this theory

In brief, I would like to state the following easy result for a PDE specialist or someone that has studied pseudodifferential operator theory.

Theorem 2.1. *Assume various kindness assumptions. Then asymptotically in $\mathbb{R}^4$, via standard theory for non-linear elliptic operators of order two, the solutions to the non-linear QFT PDE's exist.*

3. On the nature of scattering for this theory

Theorem 3.1. *Assume various kindness assumptions as in the above theorem, such as that the solution and data sections or fields of the non-linear QFT PDE's satisfy a $L^2(\mathbb{R}^D, d^Dx)$ condition, d^Dx the standard Lebesgue volume measure [15]. Then the Fourier-transform of outgoing fields or states exists at any order in perturbation theory and gives well defined cross-sections and S-matrix elements at infinity.*

4. Another derivation of the dimension counting and critical dimension 4 for independent metric and vielbein, also called simplified linear setting

Set $\mathbb{X} := \theta$ the vielbein in the matrix formulation, see Torbrand Dhrif [14]. Then normalize the metric h to simplify, so that $h = 1$, and do the scaling procedure so that $\mathbb{X}$ has length dimension -1, so $\mathbb{X}$ corresponds to e in the first section. Then

$$|\not{D}\ \mathbb{X}|^2 \sim tr(\mathbb{X}^T \Box \mathbb{X}) + \cdots .$$

[15] The case of Hilbert-Schmidt propagators and their standard L^2 Hilbert space setting is interesting with regard to this. This also has ties to renormalization theory. For example you may want to deform a pathological propagator to a Hilbert-Schmidt propagator and then analytically continue the expression you have in the deformation or renormalization parameter.

Since $\Box$ has length dimension -2, $\mathbb{X}^T$ has length dimension -1 and $\mathbb{X}$ has length dimension -1 we get total length dimension -4. Since the volume measure has length dimension 4 in dimension 4 the action is dimensionless in $D = 4$. A action where all parts are dimensionless in this way is scale invariant. Thus we see that we have critical dimension $D = 4$ in this setting.

5. Derivation of the dimension counting and arbitrary critical dimension for dynamical metric in terms of vielbeins, full non-linear situation

Now you scale both the vielbein and the metric and you realize that scaling the vielbein implies a scaling of the metric. This is the general situation. The Kroenecker delta δ_{ab} does not change. Since $g_{\mu\nu} = e^a_\mu \delta_{ab} e^b_\nu$ and $\mathbb{X} := e^a_\mu$ is normalized or scaled, so that it has physical length dimension -1, $g_{\mu\nu}$ has to have length dimension -2. Then, up to reorderings,

$$|\not{D}\ \mathbb{X}|^2 \sim tr(\mathbb{X}^T G \Box \mathbb{X}) + \cdots .$$

Since $\Box$ then has length dimension 0, $\mathbb{X}^T$ has length dimension -1 and $\mathbb{X}$ has length dimension -1, and $G := (g^{\mu\nu})$ has length dimension $+2$, we get total length dimension 0. Since the volume measure has length dimension 0 in all dimensions D when you scale the metric the action is dimensionless. We are therefore done.

Let us write this in components as well. The leading and kinematical part of the Lagrangean seen as a differential form is, up to reorderings and parentheses,

$$g^{\sigma\rho}(D_\sigma e^a_\mu)(D_\rho e^b_\nu) g^{\mu\nu} \delta_{ab} \sqrt{g} d^D x.$$

We see that the physical dimension generated by all these terms combined cancels in all positive integer dimensions D.

Remark 5.1. *You can always get the dimension calculus right by not scaling the metric $g_{\mu\nu}$ and keep it dimensionless*[16]*. Also you could pretend to view the metric and the frame as independent fields, which makes the dimension calculus trivial since you then only scale the vielbeins which are a type of frame. There is no point in keeping the frame and metric as independent fields for real, since we would then have to consider the metric dynamics as independent of the vielbeins, thus with a separate equation of motion, thus we would not gain anything. We want to do gravity out of vielbeins only, and try to gain an advantage by formulating gravity in terms of vielbeins.*

Remark 5.2. *The scaling parameter L^a of the veilbein is interesting in the usual renormalization of this theory. You can also set it to be the volume of space-time in dimension a, thus defining a natural length or time unit. Thus we have $L^a := 1$ unit of length or time. So, in these units space has volume 1, space-time has volume 1, etc. This is handy when you normalize the vielbeins as the calculus then simplifies. Of course this trick depends of the topology of space-time and can not always be used.*

[16]Notice for example that AdS metrics do not scale, which is again very interesting. I will come back to this in remark 5.3 and remark 5.4.

Remark 5.3. *It seems that the normalization of vielbeins generates spacetimes that look like the boundary of AdS spaces. Let us show this; We start by looking at the vielbein θ^a. After normalization with the 'radius' parameter u_0, instead of using L, we get $\frac{\theta^a}{u_0}$ as our new vielbein. Thus we get the metric*

$$g_{boundary} = \frac{g_{\mu\nu}}{u_0^2} dx^\mu \otimes dx^\nu$$

which is the boundary of (u becomes a variable instead of the constant u_0)

$$g_{AdS} = \frac{1}{u^2}(g_{\mu\nu} dx^\mu \otimes dx^\nu + du^2)$$

one often sets $g_{\mu\nu} = \eta_{\mu\nu}$ in the literature so this becomes

$$g_{AdS5} = \frac{1}{u^2}(\eta_{\mu\nu} dx^\mu \otimes dx^\nu + du^2)$$

where I wrote AdS5 to emphasize the dimension of the 5-dimensional AdS space. See Natssume[20], on page 176 for example, and on chapter 6 where he explains AdS spaces. I do not however think that it is necessary to use these AdS spaces in the present formalism, we can work with the usual metric and 4-dimensional spacetime in the calculus. This remark is more of an aside.

Remark 5.4. *Our spacetimes must be AdS boundaries or similar spaces, since the full metric tensors of these spaces will scale well, being invariant under scalings. That also means that the metric tensor(you can see it as a Kahler form, so I am not talking about the components but the full tensor) is dimensionless after normalization, which is imperative, since only then will the action describe a graviton with a Born interpretation. I am saying that $g_{\mu\nu}$, the metric prior to normalization in remark 5.3, and g_{AdS} are both dimensionless.*

Remark 5.5. *The calculus is tedious and complicated, and very often it simplifies by looking at the vielbeins and the metric as independent fields, as stated above. We called this the simplified linear setting, thus working around a background metric h_{ab}, which yields the dynamical metric as $g_{\mu\nu} = \theta^a_\mu h_{ab} \theta^b_\nu$. Of course this causes a linear oversimplification of these matters and does not capture the full complexity of the situation, which also is ambiguous, but does it capture enough?*

Remark 5.6. *Remark* 5.5 *on the complicated calculus is tied to the fact that you will often have to simplify the matter by assuming that the variation of the Dirac operator with respect to the vielbeins should be zero, that is $\delta \not{D} = \epsilon[A, \not{D}] = 0$, ϵ a small real number, and A generating the flow of vielbeins. This is a covariant energy conservation statement and so a natural physicality condition. You can read $\delta \not{D} = 0$ as the statement that the Dirac operator does not depend of the vielbeins over the background, so $\not{D}^2$ becomes a linear operator. If you can assume $\delta\Gamma^\mu = 0$ in the coordinate frame this would imply this assumption.*

Remark 5.7. *This remark is about a proof of a Maldacena type conjecture and gauge-gravity duality, assuming the theory presented in this article. You can use the action*

$$S_{Gravity} = \int \theta^* \not{D}_c^{\,2} \theta \sqrt{h} d^D x,$$

where h is the background metric in remark 5.5, and $\not{D}_c$ does ot depend on the vielbeins as in remark 5.6 and only on the background metric, and D_c is compatible with respect to the background metric and background vielbeins. This is to work with a linear operator that defines a functional determinant. You can also make the Taylor series approximation for the non-linear operator [17]

$$\not{D}^{\,2} = \not{D}_c^{\,2} + O(\delta\theta) + O(\delta\omega) + O(\delta A)$$

where c denotes dependence on the background fields only, and thus $\not{D}_c^{\,2}$ defines a linear operator. Here O is big ordo and δ denotes a small perturbation. Since this is so interesting I will give an example of such a computation or proof in $D = 4$ for gauge-gravity duality at 1- loop. We use the operator symmetry derived in the 1:st appendix at the end of this article. We have [18] [19] [20]

[17]Here I assume that only the operator 'Hamiltonian' $\not{D}^{\,2}$ is given and we do not look at the Lagrangean, so we are working in the operator formalism. The background connection $D_{\mu c}$ that we work with is still compatible with respect to the background.

[18]To get these identities to work you will have to realize that the graviton has double the gravitational charge of the gravitational charge of the Weyl fermion. This means that $\frac{g_G}{2} = g_F$, where g_G is the gravitational charge of the graviton and g_F is the gravitational charge of the Weyl fermion. So you will have to insert a factor $\frac{1}{2}$ in the gravity connection. Of course this is connected to the spin of the graviton, that has spin 1, as compared to a Weyl fermion which has spin 1/2. Thus I suspect that the gravity quantum is given by natural integer multiples of 1/2, as given by the spin. Also, if we for example set $g := \sqrt{4\pi G} M_0$ a gravitational coupling constant with M_0 a fundamental mass, a Regge slope will arise from the spin parameter. Of course this g is dimensionless as required.

[19]The graviboson $\not{\omega}$ term in the vielbein Dirac operator differs from the graviboson term in the spin Dirac operator by a factor 1/2, so $\not{\omega}_G/2 = \not{\omega}_F$, $\not{\omega}_G$ the graviboson term in the vielbein Dirac operator and $\not{\omega}_F$ the graviboson term in the spin Dirac operator. I absorbed that in the coupling constant in the previous footnote.

[20]There is a pitfall that you absolutely have to notice, although it is true that the square Dirac operators will coincide, also the domains or Hilbert spaces that these square Dirac operators act on have to coincide, of course up to the representation used. If we set $\not{D}^{\,2} : H \mapsto H$ as the square Dirac operator, H must be the same for the two different Dirac operators. This issue was neglected by me in previous work. In $D = 4$ we have that $F\mathbb{C}^4$, denoting fermionized $\mathbb{C}^4$, is both spinor space and the complexified vielbeins over a point in space-time so we come out fine. More explictly, we work with the space $L^2\Gamma^\infty(M^4, F\mathbb{C}^4) \subset H$ of smooth square integrable sections of a bundle over space-time with fiber $F\mathbb{C}^4$. Since $[\Gamma^a, \Gamma^b]/4 = \theta^a\theta^{*b}$ are the complex $D \times D$ matrices, $F\mathbb{C}^4$ is the appropriate representation module for commutators of Dirac matrices in $D = 4$, both in the vielbein formalism and the spinor formalism. Also, only these commutators appear acting on spinor space or vielbein space for the square Dirac operators so we only need a representation module for these commutators, which is a simplification of the situation. To be even clearer the vielbein connection is $D_{\mu,V} = \partial_\mu + \omega_\mu^\alpha T_\alpha + A_\mu = \partial_\mu - \omega_{\mu ab}\theta^a\theta^{*b} + A_\mu = \partial_\mu - \omega_{\mu ab}\frac{[\Gamma^a,\Gamma^b]}{4} + A_\mu$, and the spinor connection is $D_{\mu,S} = \partial_\mu - \frac{\omega_{\mu ab}\theta^a\theta^{*b}}{2} + A_\mu = \partial_\mu - \frac{\omega_{\mu ab}}{2}\frac{[\Gamma^a,\Gamma^b]}{4} + A_\mu$, so we have the same representation module for the connections. Note that $\omega_{\mu ab}$ is antisymmetric in a and b. Since only these connections appear in the connection Laplaceans $\Box$, and the rest of the terms in the Bochner identities are the same, our reasoning is true if you make a partial integration in the action for the connection Laplacean term. Only in $D = 4$ will the representation theory coincide in this manner, with the correct constant multiplying

$$Z_{Gravity} = det((i\not{D}\ )_c^2) = det((i\not{D}\ )^2) = |det(i\not{D}\ )|^2 = |Z_{GaugeTheory}|^2.$$

I used the identities derived in this article above, and I used the standard formula $det(AB) = det(A)det(B)$*, where* A *and* B *are linear operators. Of course there are Hardy-Ramunjan identities supressed above that permit us to state the bosonic gravity partition function on the left as this determinant, something that states that the partition function comes out equivalently in the bosonic (NS) and fermionic (R) formulation. On the left we use what I call the vielbein Dirac operator and on the right we use the spinor Dirac operator on the background. The square of these two operators coincides in* $D = 4$*. We can also prove this with operator traces*

$$Z_{Gravity} = Tr(e^{-(i\not{D}\ )_c^2\beta}) = Tr(q^{(i\not{D}\ )_c^2}) = det((i\not{D}\ )^2) = |Z_{GaugeTheory}|^2.$$

Here we have the linearized vielbein Dirac operator on the left and we have the spinor Dirac operator on the background on the right. $\beta \geq 0$ *is a real parameter.* q *is defined as* $q = e^{-\beta}$*. All of these identities have to be normalized, and renormalized as well, since these determinants and traces are usually ill defined.*

Now I will show a example of a Maldacena theorem with the assumptions in this article, just to make the reader aware of the applications of this theory. Assume X *and* Y *have* 4 *kinematical dimensions each but really both* 5 *dimensions, so that the total cartesian product* $X \times Y$ *has dimension* 10*. You can think of standard features of Green-Schwarz strings if you want to motivate this. Then assume that the physics of* X *and* Y *are independent in the sense that* $Z_{X\times Y} = Z_X Z_Y$*. Then we have that the supergravity proposed gives*

$$Z_{SUGRA,X\times Y,D=10} = det((i\not{D}\ )_c^2)_{X\times Y} = det((i\not{D}\ )_c^2)_X det((i\not{D}\ )_c^2)_Y.$$

Using the previous calculations in this remark we easily get

$$Z_{SUGRA,X\times Y,D=10} = Z_{Gravity,X}|Z_{GaugeTheory,Y}|^2.$$

For more calculations like these see Torbrand Dhrif[7].

Remark 5.8. *This theory is amenable to particle physics, with the standard assumptions of particle physics in* $\mathbb{R}^D$*. I expect drastic simplifications for that situation, resembling the situation for standard gauge theory.*

Remark 5.9. *The action of this theory has a* $SO(D)$ *symmetry group in the ON frame or flat space-time*[21]*.The unit determinant comes from that probabilities have to be positive. That gravity must have a* $SO(D)$ *symmetry group should come as no surprise, for example from relativity considerations. In correct signature this is* $SO(1, D-1)$ *symmetry where* D *is the dimension of space-time. Since the critical dimension is* $D = 4$ *for our model we set the symmetry group as* $SO(1, 3)$*. One also sees the need for orientable space-times to make the probability interpretation in the first section.*

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the Ricci scalar term. See the 2:nd appendix for a second proof of this theorem, but in another setting where we work with the fields θ^a as fermions instead of ψ.

[21] Actually the gauge theory part has a *spin*(D) symmetry group, so *spin*$(1, D-1)$ in correct signature.

Remark 5.10. *The pure gravity part of the Lagrangean has to be modified with a factor* $1/2$ *to get the simplest Feynman rules, and also for other reasons, such as most physical computations. So*

$$|\not{D}\ \theta|^2 \mapsto \frac{|\not{D}\ \theta|^2}{2}.$$

Remark 5.11. *The test of naive renormalizability below is the simplest test in particle physics for the validity of a Lagrangean. If a theory is not naively renormalizable you can discard it straight away from a physical point of view. Also, experience from particle physics shows that theories that are not naively renormalizable usually behave badly, and this is purely heuristic. After all, you can not expect a action that has non-vanishing length or mass dimension to be exponentiated from a physical point of view.*

Definition 5.1. *A theory is naively renormalizable if it has dimensionless action* [22] *and no dimensionfull constants.*[23] *A naively renormalizable theory is a conformal field theory (CFT)* [24] *and a CFT is naively renormalizable*[25]*.*

Definition 5.2. *A theory is a CFT(conformal field theory) if it has dimensionless action, no dimensionfull constants and a scale invariant action.*

Definition 5.3. *A theory has critical dimension* D *if it has a dimensionless action in that dimension* D*, no dimensionfull constants and the action is scale invariant in that dimension* D*. Alternatively a theory has critical dimension* D *if it is a CFT in that dimension* D*.*

Remark 5.12. *What if we do not scale the vielbein? Then it is hard to interpret the graviton, that is the vielbein, as a field tied to a probability density. You also then seem to end up in standard string theory. But,and here is something really interesting; Then you do not have to invent a new dimensionless coupling constant and the good old gravity coupling constant* G *will do, and you will normalize the pure gravity action by* $8\pi G$*, this including the remark 5.10 on Feynman rules. For example my calculation of Gauge-Gravity duality in remark 5.7 will carry through. So, even if some people may not agree with this normalization of the vielbeins in this article it still has great value for them. After all this proof is so elegant that few people could object.*

Remark 5.13. *We can also define scalings for vielbeins and other fields, without a probability interpretation. See Natsuume[20] for example, on page 68 to page 69. He sees such scalings as conformal transformations, which is very close as an idea.*

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[22]When I write dimensionless or dimensionfull I usually mean it in the sense of length dimension or mass dimension. We scale fields and not constants here.

[23]This implies scale invariance for such a scenario. This is not the case if there are dimensionfull constants involved.

[24]As an example of dimensionfull constants I mention these; A mass term such as in the Dirac spinor mass term, a massive ϕ^4-scalar or the Hilbert-Einstein gravity coupling G. These usually break conformal symmetry and screw up the scaling properties. The Dirac fermion mass and ϕ^4-scalar mass are not real villains, but the gravity coupling usually presents a problem. I choose to define these things in this way to make the construction as elegant and coherent as possible.

[25]At least by my definition.

Remark 5.14. *There is a notion that I call partial conformal invariance. Basically you do not have to scale all independent fields in a action, often you can scale just and normalize just some of the fields. The action* L_{total} *above contains many fields, and when you do regular CFT you can refrain from normalizing the metric and end up immediately in critical dimension* $D = 4$ *as in the first derivation of critical dimension in this article, see sections* 1 *and* 4*. If you scale all fields you will end up in arbitrary critical dimension for the action* L_{total}*. And, since you are at liberty to choose dimension you just pick* $D = 4$ *for our physical situation. So what happens is that you want the action to behave well in the critical dimension predicted by doing different combinations of field scalings. For our action that means that we are working in* $D = 4$*.*

Remark 5.15. *For the full nonlinear theory we must work with a background field gauge, or at least with a expansion around the background, since the theory otherwise only contains vertices and no propagators. This is a important remark.*

Remark 5.16. *Earlier in this article I stated in a footnote that one has to define a dimensionless coupling constant* g *for gravity, instead of the usual gravity coupling constant which actually has physical length dimension. The other coupling constants of physics for electromagnetism, electroweak theory and chromodynamics do not have dimension and we want to treat all theories or sectors in the same manner. The gravity constant* G *has such dimension that the usual Hilbert-Einstein action becomes of correct dimension in* $D = 4$ *and is therefore dimensionfull. The gravity constant* g *that we use microscopically differs from the macroscopic gravity constant* G*.*

Remark 5.17. *The gravitational part of this theory looks like a gauge theory. This makes it more probable as a viable theory, since the other interactions of physics are accurately described by gauge theories, that we for example believe are renormalizable.*

Remark 5.18. *In my early work I had the idea that gravity and space-time would be accurately described by hypercomplex numbers, also called Clifford algebras or Dirac algebras. These ideas can be traced to the work of Hamilton, who discovered the quaternions, and are very old. Since* $\Gamma^a = \theta^a + \theta^{*a}$ *we see a clear relationship between gravity (the vielbeins* θ^a*) and hypercomplex generators* Γ^a*. So basically space-time is locally a (possibly pathological) hypercomplex manifold. Of course there is not anything very new to this, but it could be a good philosophical point.*

Remark 5.19. *The good way of defining the partition function* $Z_{Gravity}(i\not{D})$*,* $\not{D}$ *here the vielbein Dirac operator, is* $Z_{Gravity}(i\not{D}) := \sqrt{Z_{Gravity}((i\not{D})^2)}$*. This has to do with matters of the spectrum of this operator.*

6. What is a quantization of a theory ?

Let us look at the examples of QED, electroweak theory and QCD. They are not statements or proofs of mathematics, they are statements of theoretical physics, with all they entail such as the philosophy of physics (that statements are true until proved wrong in some sense, either theoretically or experimentally). No one has reached further for the theories [26] of QED, electroweak theory and QCD other than calculations at finite loop order and applying certain dualities to probe these theories. This is of course in $D = 4$. This very same philosophy of physics applies to the gravity theory given above. It is a triviality that this gravity theory is ok to 1-loop order, for example if you use a functional determinant, with some renormalization scheme such as zeta function renormalization, and differentiate with respect to background fields in a background field gauge. I claim that the simplified linear theory lives in the sense of naive renormalizability in dimension 4 and that the full nonlinear theory seems to live in arbitrary critical dimension, and that the interaction term of this nonlinear theory is Einstein gravity. There is no reason not to believe that this theory behaves well at arbitrary finite loop order.

7. Appendix 1; Calculating the Bochner formula and the Ricci term

This section is about the Bochner formula for the fully non-linear theory. We will not assume that the connection is compatible with the vielbeins in this calculation, so we are working with what we call a off-shell connection, and we start in a arbitrary frame. Let us start of with the trivial calculation

$$\not{D}^{\,2} = \Gamma^\mu D_\mu \Gamma^\nu D_\nu = \mu + \Gamma^\mu \Gamma^\nu D_\mu D_\nu$$

where we define [27]

$$\mu := \Gamma^\mu (D_\mu \Gamma^\nu) D_\nu = \Gamma^\mu (\partial_\mu \Gamma^\nu) D_\nu + \Gamma^\mu \Gamma^\rho \omega^\nu_{\mu\rho} D_\nu = \Gamma^\mu (\partial_\mu \Gamma^\nu + \omega^\nu_{\mu\rho} \Gamma^\rho) D_\nu.$$

Assume for simplicity Cartesian coordinates and a flat metric and let us use the following calculation from Nakahara[18] on page 440;

$$\Gamma^\mu \Gamma^\nu D_\mu D_\nu = [g^{\mu\nu} + \frac{1}{2}[\Gamma^\mu, \Gamma^\nu]]\frac{1}{2}[\{D_\mu, D_\nu\} + F_{\mu\nu}] = D^\mu D_\mu + \frac{1}{4}[\Gamma^\mu, \Gamma^\nu] F_{\mu\nu}.$$

Let us then set

$$\Box := D^\mu D_\mu + \mu.$$

In general coordinates and metric this is

$$\Box := \frac{1}{\sqrt{g}} D_\mu \sqrt{g} g^{\mu\nu} D_\nu + \mu.$$

Thus we have for the square of the Dirac operator

$$\not{D}^{\,2} = \Box + \frac{1}{4}[\Gamma^\mu, \Gamma^\nu] F_{\mu\nu}.$$

[26] You should omit the mass terms of these theories here in this article. This is a first analysis; On a compact background space-time the mass terms become compact deformations that will not affect the situation very much. Of course the physical numerical predictions change, but the niceness will be conserved.

[27] This term μ vanishes in the ON-frame for a compatible or on-shell connection by virtue of the first Cartan structure equation. You then get the usual Bochner identities. It is linked to the torsion of the connection.

Writing, say in the ON-frame,

$$\not F = \frac{1}{4}[\Gamma^a, \Gamma^b]F_{ab},$$

this becomes

$$\not D^2 = \Box + \not F .$$

Of course if you set the gauge group as $SO(D) \times U(N)$ this is

$$\not D^2 = \Box + \not R + \not F .$$

We thus similarly define

$$\not R = \frac{1}{4}[\Gamma^a, \Gamma^b]R_{ab}.$$

We now show

$$\theta^* \not R \theta = \theta^* R^{\alpha}_{ab} T_\alpha \theta^a \theta^{*b} \theta.$$

We have for the commutators of the Clifford or Dirac algebra generators

$$[\Gamma^a, \Gamma^b] = [\theta^a, \theta^b] + [\theta^{*a}, \theta^{*b}] + 2(\theta^a \theta^{*b} - \theta^b \theta^{*a}).$$

Thus keeping terms that do not vanish because of the exterior algebra in

$$\theta^* \not R \theta$$

we get the above formula by antisymmetry in the last two of the indices in R^a_{bcd}, the indices c and d;

$$\theta^* \not R \theta = \theta^* R^{\alpha}_{cd} T_\alpha \frac{[\Gamma^c, \Gamma^d]}{4} \theta = \theta^* \frac{R^{\alpha}_{cd} T_\alpha}{2} (\theta^c \theta^{*d} - \theta^d \theta^{*c})\theta = \theta^* R^{\alpha}_{cd} T_\alpha \theta^c \theta^{*d} \theta.$$

We use the inner product for vector valued forms as

$$< \eta, \zeta >= \eta^* \zeta = *((*\eta)^\dagger \wedge \zeta).$$

$*$ is the Hodge star and $\dagger$ is the Hermitian adjoint. We proceed to the calculation of the Ricci term

$$\theta^* \not R \theta.$$

First of all

$$*\theta^* \not R \theta = *([*\theta^a]^T [R^{\alpha}_{bc} T_\alpha \theta^b \theta^{c*}][\theta^d]) = *((*\theta_a) \wedge R^a_{bcd} \theta^c \delta^{db}).$$

Because of antisymmetry in c and d this becomes

$$- * ((*\theta_a) \wedge R^a_{bdc} \theta^c \delta^{db}) = - * ((*\theta_a) \wedge Ric^a_c \theta^c) = - * ((*\theta^a) \wedge Ric_{ac} \theta^c)$$

which is

$$= *(\delta^{ac} Ric_{ac})(\epsilon_M) = Ricci (= Ricci \cdot N)$$

where ϵ_M is the volume and Levi-Cevita form on M, and we desuppress a gauge group factor in the parenthesis, where N is the number of colors.

This color term N can be taken care of by a suitable normalization of the colored vielbeins if one wants to. We set the following convention

$$\epsilon^{123\cdots D} = 1.$$

We thus have in $D = 4$ on vierbeins the Bochner reminiscent formula

$$\theta^* \not{D}^2 \theta = \theta^* (\Box + \frac{Ricci}{4} + \not{F})\theta.$$

This looks similar to the spinorial Dirac operator identity that we started this article with. So, as the reader understands, I see a possible supersymmetry in $D = 4$ [28]. This identity for the square vielbein Dirac operator $\not{D}^2$ generalizes to arbitrary dimension D as

$$\theta^* \not{D}^2 \theta = \theta^* (\Box + \frac{Ricci}{D} + \not{F})\theta.$$

Again, I remind the reader that we worked with the fully non-linear theory in this section.

8. Appendix 2; Another proof of gauge-gravity duality

In this version of gauge-gravity duality we assume that we work in arbitrary D, and that θ^a is linked to a fermion if we change it's transformation properties. We assume that we have a linearized vielbein Dirac operator available, since we want to make sense of the functional determinants we use in this section. We start with the following identity

$$Z_{Gravity} = det((i\not{D})_c^2) = |det((i\not{D}_c))|^2.$$

Since we have the identity $\not{D}_c|_{\omega:=\omega/2} = \not{D}$, where $\not{D}_c$ is the linearized vielbein Dirac operator and $\not{D}$ is the spinor Dirac operator, and both act on the space of smooth square integrable sections of a bundle over space-time with fiber $F\mathbb{C}^D$, that is fermionized $\mathbb{C}^D$, which we denote as $L^2\Gamma^\infty(M^D, F\mathbb{C}^D) \subset H$, we get that [29]

$$Z_{Gravity} = det((i\not{D})_c^2) = |det((i\not{D}_c))|^2 = |det(i\not{D})|^2 = |Z_{GaugeTheory}|^2.$$

The key point here is to realize that the domain of the vielbein Dirac operator coincides with the domain of the spinor or usual Dirac operator (actually these domains are even equivalent). This is my version of gauge-gravity duality and it is correct if we assume the ideas in this article. The reader may want to make the calculation

$$\not{D}_c|_{\omega:=\omega/2}\theta = \not{D}\theta$$

[28] Note again that there is a factor 1/2 that relates the gravitational connections. If the gravitational connection vanishes(so that $\omega = 0$) this equivalence of square Dirac operators, between the square vielbein Dirac operator and the square spinor Dirac operator, is a trivial identity as even the Dirac operators coincide, then not including the fact that the Hilbert space modules may differ. This is, as stated again, a clear sign that we are doing something right.

[29] I here suppress a factor 1/2.

to check this, where θ is the vielbein as a vector valued 1-form in E. Cartan's usual notation. [30] You can easily see that the vielbein θ coincides with the fermion field in all D when you change transformation properties, that is remember the factor 1/2, which is related to how 1/2-spin fields transform under rotations and boosts. For the sake of completeness I state the spinor Dirac operator on the background [31]as

$$\not{D} = \Gamma^\mu(\partial_\mu - \frac{\omega_{\mu ab}}{2}\frac{[\Gamma^a,\Gamma^b]}{4} + A_\mu) = \Gamma^\mu(\partial_\mu - \frac{\omega_{\mu ab}}{2}\theta^a\theta^{*b} + A_\mu)$$

and the linearized vielbein Dirac operator on the background is

$$\not{D} = \not{D}_c = \Gamma^\mu(\partial_\mu + \omega^\alpha_\mu T_\alpha + A_\mu) = \Gamma^\mu(\partial_\mu - \omega_{\mu ab}\theta^a\theta^{*b} + A_\mu),$$

where T^α are the generators of the Lie algebra $\mathfrak{so}(1, D-1)$ in the fundamental representation, if we use a space-time with correct signature. Also, I use the following Dirac or Clifford algebra representation,

$$\Gamma^a = \theta^a + \theta^{*a},$$

in this article, θ^a here the vielbeins or the fermions. The factor 1/2 that pops up above is related to the local difference between $SO(1, D-1)$ and $spin(1, D-1)$[32]. We may want to call this $\mathcal{N} = D$ duality, after kind suggestion.

This proof is different from the proof of gauge-gravity for vielbeins $\theta = (\theta^a)$ and fermions ψ in remark 5.7. The big difference is that the gauge-gravity duality in remark 5.7 is valid in $D = 4$, where as the proof in this appendix is valid in all D. The reason is that these two fermion fields are not the same.

You may want to transform the fermions, and vielbeins, θ^a under $SO(1, D-1)$, just like $\Gamma^a = \theta^a + \theta^{*a}$, the gauge-gravity duality then obtained is a triviality in all positive integer dimensions D. This last way of looking at fermions makes the duality that we state both trivial and profound. This last third duality is probably the best.

Also, you may want to look on Green, Schwarz and Witten[19], appendix 5.A, volume I, for properties of the groups $SO(2n)$ in such representations, n a positive integer.

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[30]Here, again, $\not{D}_c$ is the linearized vielbein Dirac operator and $\not{D}$ is the spinor Dirac operator.

[31]Of course you do the relevant computations in the fully non-linear situation to prove various identities, such as the identities for the two Dirac operators below, and then expand the relevant Dirac operators around the background as a last step, only keeping the background term. For the linearized Dirac operators on the upper part of this page you should use $\omega = \omega^\alpha T_\alpha$ and $\omega/2$ as the appropriate connections for the vielbein connections and spinor connections respectively. That this makes sense is apparent from the Dirac operator identities on this page.

[32]I am not in any way stating that $spin(1, D-1)$ and $SO(1, D-1)$ are equivalent as groups or manifolds. In fact they are usually different.

9. Appendix 3; The Maldacena/Gauge-Gravity type conjectures in this article

There are three Maldacena/Gauge-Gravity type conjectures in this article. These conjectures are generalizations of early conjectures treated by among others Maldacena and Polchinski that I first saw ca 1996-1997, and should not be confused with other conjectures, and are related to the concept of holography.

(1) The 1:st conjecture is that there is a duality between the vielbeins θ^a on space-time that describe gravity and the Clifford or Dirac algebra generators Γ^a that describe fermions on space-time. This is completely evident in all dimensions by $\Gamma^a = \theta^a + \theta^{*a}$, although there may be finer points such as a non-linear Dirac equation. In fact we could describe the fermions directly in terms of the fields $\theta^a = \theta^a_\mu dx^\mu$.
(2) The 2:nd conjecture is a duality between fermions described by Dirac spinors ψ and gravity described by the vielbeins θ in dimension $D = 4$. We sketched on a proof in remark 5.7, above all in footnote 20. We know that the dimension of the spinors and vielbeins coincides in dimension $D = 4$, and we started on a proof. We need to inspect this proof and make it with modern mathematical standards of rigor.
(3) The 3:rd conjecture and last Gauge-Gravity type conjecture is what we called $\mathcal{N} = D$ duality in Appendix 2. I do not know if that conjecture is true.

Again, this work should not be confused with other conjectures by e.g Maldacena, Polchinski and other authors. The conjecture I mention here is that they wanted the factorization

$$Z_{IIA} = Z_{Gravity,X} Z^2_{GaugeTheory,Y},$$

where $X = AdS^n$ and $Y = S^m$. Here AdS^n means a anti-deSitter space and S^m a sphere. I think that X and Y could be more general, that is the point of this matter. There is a strong relation between Gauge-Gravity duality and Maldacena type statements, they are very simply related. With Gauge-Gravity I usually mean statements of the type

$$Z_{Gravity} = Z_{GaugeTheory} \cdot Z_{GaugeTheory} = Z^2_{GaugeTheory}.$$

10. Appendix 4; The Lagrangean for the non-linear theory

The non-linear theory Lagrangean could be described by

$$L_{total} = (\theta^* \Box \theta + \bar{\psi} i \not{D} \ \psi + \frac{1}{4}|R+F|^2 + Ricci \cdot N + \theta^* \not{F} \ \theta)\sqrt{g} d^D x,$$

where the volume form is $\sqrt{g} d^D x = \theta^1 \wedge \cdots \wedge \theta^D$. N is the number of colors, but for $N = 0$ this becomes

$$L_{total} = (\theta^* \Box \theta + \bar{\psi} i \not{D} \ \psi + \frac{1}{4}|R|^2 + Ricci)\sqrt{g} d^D x.$$

Again, do not forget that we have redefined the coupling constant to a dimensionless coupling constant, so the omission of the standard factor $-16\pi G$ is not wrong. Remember that we define

$$\Box := \frac{1}{\sqrt{g}} D_\mu \sqrt{g} g^{\mu\nu} D_\nu + \mu.$$

Here μ is a torsion related term. Let us define the torsion related term

$$\Xi := \theta^* \mu \theta = \theta^* \Gamma^\mu (\partial_\mu \Gamma^\nu + \omega^\nu_{\mu\rho} \Gamma^\rho) D_\nu \theta$$

and

$$\Delta := \frac{1}{\sqrt{g}} D_\mu \sqrt{g} g^{\mu\nu} D_\nu.$$

Then the full non-linear theory Lagrangean reduced to simpler components is

$$L_{total} = (\theta^*(-\Delta)\theta - \Xi + \bar{\psi} i \not{D}\ \psi + \frac{1}{4}|R+F|^2 + Ricci \cdot N - \theta^* \not{F}\ \theta)\sqrt{g} d^D x.$$

Here I have desuppressed all involved minus signs, see footnote 6. For $N = 0$ this is

$$L_{total} = (\theta^*(-\Delta)\theta - \Xi + \bar{\psi} i \not{D}\ \psi + \frac{1}{4}|R|^2 + Ricci)\sqrt{g} d^D x,$$

again with all minus signs desuppressed. Of course we have here assumed that the graviton and Weyl fermion have the same color gauge group $U(N)$. For the other, but still very interesting, case that the Weyl fermion has color gauge group $U(N)$ while the graviton has no color gauge group at all the Lagrangean becomes

$$L_{total} = (\theta^*(-\Delta)\theta - \Xi + \bar{\psi} i \not{D}\ \psi + \frac{1}{4}|R+F|^2 + Ricci)\sqrt{g} d^D x.$$

All of these Lagrangeans are before normalization, and before division by the factor 2 that gives better Feynman rules, as in remark 5.10.

11. Appendix 5; The Lagrangean for The Simplified Theory

The Lagrangean could be described, ex normalization, by

$$L_{total,simplified} = (|\not{D}\ \theta|^2 + \bar{\psi} i \not{D}\ \psi + \frac{1}{4}|R+F|^2)_h \sqrt{h} d^D x.$$

Here h is a fixed background metric. This theory has the property that the equation of motion derived by variation of the dynamical vielbein over a fixed background, for say $h_{\mu\nu} := \eta_{\mu\nu}$ a flat background, is the usual first Cartan structure equation with color, for the situation with no torsion. Thus this becomes

$$d\theta + (\omega + A) \wedge \theta = T = 0.$$

This theory is a CFT in $D = 4$, and could give a 'accurate enough' theory for particle physics, without going into if it is a correct theory or not. This Lagrangean is before division with a factor 2 that gives better Feynman rules as in remark 5.10.

12. Appendix 6; The Dimensionless Coupling Constant

We define

$$g := \sqrt{4\pi G} s M_0,$$

M_0 a fundamental mass, and $s \in \mathbb{N}/2$ a half integer spin parameter. This coupling constant enters the calculations by setting

$$T_\alpha \mapsto gT_\alpha.$$

More generally

$$\omega \mapsto g\omega.$$

13. Appendix 7; The Colored Vielbein

The colored vielbein can be described by

$$\theta := [\theta^a \otimes |i>].$$

Here θ^a is the usual vielbein of Cartan's formalism, and $|i>$, $i \in \{1, \cdots, N\}$, are ON basis sections on a complex vector bundle over space-time, the fiber being the module for the defining representation of $U(N)$.

14. Conclusions and future directions

Let us comment on the fully non-linear theory. This Lagrangean(density) or theory must be thoroughly investigated without ad hoc assumptions, such as remark 5.5 and 5.6. I believe that this theory may be correct or a at least is a good start, so far, but it probably has to be checked via computer software since the calculations are so tedious in the general situation. I do not make any pretense at having done exhaustive calculations or proofs, I am just proposing a theory. Also, it is not at all trivial if gravity can be described by vielbeins, after all the ON-connection is not at all the Levi-Cevita connection, one should not confuse one for the other, so here is a ad hoc assumption. I expect drastic simplifications for the particle physics scenario of a background field expansion around flat $\mathbb{R}^D$, $D = 4$, and that is a good and motivated direction of future research. This theory has siblings in the literature, such as in Polchinski[17], and Witten et al [8]. The difference is the Einstein interaction term that does not have to be added, and comes naturally, and that it lives in dimension 4, in fact all dimensions. I also think that the simplified linear theory should be thoroughly investigated since it is a obvious CFT only in $D = 4$. Also, it would be very good if someone, a representation theorist, could look a little closer at gauge-gravity for vielbeins θ and spinor Dirac fields ψ in these theories. The relation between the simplified theory and the non-linear theory has to be investigated explictly. Specifically I would like to have a proof of that the simplified theory is a approximation to the non-linear theory.

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