On Probability of a Stock Crash, Credit Risk and Default

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1. Abstract

In this paper we compute the risk neutral intensity, hence probabilities(risk neutral and physical), for a stock crash occurrence. We also give the solution of the American call option in a perpetual defaultable setting. The values of perpetual American calls in a defaultable setting are also computed for two standard models, and this is a model for implied default intensity.

2. On The Probability of a Stock Crash and The Perpetual American Call in a Defaultable Setting

Intensity based models are common in the litterature, see e.g. P.J. Schonbucher[7], Duffie and Singelton[3] or Cossin and Pirotte[2]. Usually these models are investigated via stochastic methods, something that we in the present paper defer from doing, instead reducing these models to a free boundary PDE approach. This results in a formula for the default intensity of a stock market, something that we see as characterizing the probability of a crash on such an exchange. A stock crash is characterized here in the beginning of the article by that no company in the given sector we investigate can honor all of its commitments. Later we make an ansatz at a more refined definition.

3. The Black-Scholes Model

We assume that the underlying variable(the stock market capitalization) follows a one-dimensional geometric Brownian motion. Thus we have that

Hypothesis 3.1.

$$dX_t = rX_t dt + \sigma X_t dB_t$$

where B_t is Brownian motion, r is the short rate and σ is the volatility of X_t, defined here by $\sigma(x,t)^2 X_t^2 dt = Var(dX_t)$. The relevant probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{t \le s \le T}, P)$ is as follows: It is the canonical Wiener space where Ω is the space of continuous functions $C([t,T],\mathbb{R})$, $(\mathcal{F}_s^0)$ is the filtration generated by the coordinate process $B_t(\omega)$, $\omega \in \Omega$, P is Wiener measure on $\mathcal{F}_T^0$, $\mathcal{F}$ is the P-completion of $\mathcal{F}_T^0$,and for each s, $\mathcal{F}_s$ is $\mathcal{F}_s^0$ completed with the null sets of $\mathcal{F}$.

Here we have also

Hypothesis 3.2. $r \ge 0$, $f \ge 0$.

The defaultable Black-Scholes equation, is then

$$(\partial_t + \frac{\sigma^2 x^2}{2}\partial_x^2 + rx\partial_x)F(t,x) = (r+f)F(t,x)$$
$$F(T,x) = g(x).$$

on the domain $t \in [0,T], x \in \mathbb{R}_+$, see Duffie(1996), section F, Chapter 5, and BjÃ¶rk(1998), Proposition 10.5. We also assume the following hypothesis:

Hypothesis 3.3. *Here g(x) is at most of polynomial growth and is continuous. The solution is required to be C^1(it only depends on space in view of perpetuality) on the interior of the domain.*

4. Main Results

We begin by stating the solution to the perpetual American call problem.

Proposition 4.1. *The value of a perpetual American call with strike $K > 0$ is given by*

$$F(x) = (b-K)(\frac{x}{b})^{\alpha_1}$$

$$b = \frac{K\alpha_1}{\alpha_1 - 1}$$

$$\alpha_1 = \frac{1}{\sigma^2}(-(r-\frac{\sigma^2}{2}) + \sqrt{(r+\sigma^2/2)^2 + 2\sigma^2 f}).$$

Here b is the optimal stopping boundary.

Proof. Here is a constructive proof. This perpetual American call option problem can be characterized by the following free boundary value problem, where smooth fit gives the boundary:

$$(\frac{\sigma^2 x^2}{2}\partial_x^2 + rx\partial_x)F(x) = (r+f)F(x)$$
$$F(b) = (K-b)^+$$
$$(\partial_x F)(b) = 1$$

Here we require the solution to be bounded at the origin. Solving this Euler type differential equation with an Euler ansatz $F = C_1 x^{\alpha_1} + C_2 x^{\alpha_2}$ and discrading the singular term at the origin that comes from the negative root of the indicial equation we obtain the solution. □

It is worth while noting that when we have small default intensities we can extract revealing asymptotic behaviour.

Corollary 4.1. *As f tends to zero we have*

$$b \sim K(\frac{r+\sigma^2/2+f}{f})$$
$$\alpha_1 \sim 1+\frac{f}{r+\frac{\sigma^2}{2}}$$

From these last asymptotics we see that the value of the optimal exercise boundary becomes bigger and bigger in stock space as the default probability goes to zero.

Let us use the following model to simulate a stock market:

- The investors have debts, to finance their investments, with face value of debt K and present value of debt D determined by the market.
- The value of this market, say the Swedish financial markets, can be modelled by a contract function $g=(x-K)^+$ where x is the total capitalization of the market. This contract is subject to default of a possibly larger market and can be excercised at any time(this latter feature is a drastic simplification, in reality it can only be partially excercised). Absolute priority holds in this stock market so debt holders are reimbursed before stock holders receive their gains. We also assume that debts are constant over time.
- Markets are frictionless, so there are no transaction costs or taxes. Asset values are continuous as opposed to discrete and are traded continuously.
- Shorting of assets is allowed, and bid-ask spreads are null. Furthermore lending rates are equal to borrowing rates.
- There is a bank account whose rate of return is known.
- The stock market capitalization follows a geometric Brownian motion as specified above.
- Management acts in a way to maximize company/shareholder value.
- There are no dividends. This is a simplifying assumption and can possibly lead to inconsistencies over an infinite time horizon, but is remedied by the substitution $r:=r-\delta$, $f:=f+\delta$ where δ models the dividends.
- Default of this stock market(or rather submarket), which we say is a stock crash,is an abstract event which implies that no company in this subsector honors its commitments. Our model only investigates this submarket. We assume that the commitments must be honored at some finite time so that the optimal stopping time pertaining to the problem is a.s. finite. Default of one company correlates perfectly with default of other companies since they are assumed to be of roughly the same kind(

this is a feature that gives a robustness of the model but is also a drastic simplification).

Under these assumptions we directly obtain the following corollary

Corollary 4.2. *The default intensity is given implicitly, i.e is backed out by*

$$F(f, K, r, \sigma) = x - D$$

for any $x \geq D$ and constant rates and volatilities. This means that one inverts the formula for a given market value, face value debt K, rate r, volatility σ, and present debt value D.

Remark 4.1. *The existence of indices makes the ascertainment of the model volatility relatively easy.*

Assume instead that we model the value with a square-root CEV model, thus modeling

$$dX_t = (rX_t)dt + \sigma\sqrt{X_t}dB_t.$$

The solution to the appropriate free boundary problem can then be obtained by a Frobeinus expansion as a linear superposition of the functions $p_i = x^{\lambda_i} \sum_{m \geq 0} a_{i,m} x^m$, $i \in \{1, 2\}$ where $a_{i,0} = 1$.

$$\lambda_1 = 1, \lambda_2 = 0, a_{i,m+1} = -\frac{2(r(\lambda_i + m) - (r + f)}{\sigma^2(\lambda_i + m + 1)(\lambda_i + m)}$$

The function p_2 is discarded since it is non-zero at the origin, but is interesting to model a put. Thus $F = ap_1(x)$, where $a = (b-K)/p_1(b)$. $b > K$ is then numerically solved for by solving $F'(b) = ap_1'(x) = 1$ (this is usually difficult to solve analytically as this requires finding the roots of a polynomial of degree higher than four.)

Example 4.1. *If we set the value of the parameters at $x = 10MSEK$, $\sigma = 0.3$, $r = 0.03$, $K = 4.5MSEK$, $D := 1MSEK$ we obtain a default intensity of $f = 0.00212$. We are calculating the default intensity of the company from the shareholders perspective, so decreased debt leads to decreased intensity. This is counterintuitive at first glance, but is correct once we recognize that the debt valued on the 'wrong' side, that is on the debt-holders side. The debt from the debt markets perspective is $4.5 * 4.5 = 20.25$ in the above example.*

Example 4.2. *We may want to try the Ericsson company as a trial of our default model. Ericsson had big financial problems during 2001-2002 and remedied these problems later. This should show in the default probability. Indeed, we have with data taken from the annual reviews that $f = 0.01$ for may 2005 where as $f = 0.16$ for september-october 2002. Thus the 1-year default probability was roughly 16% for Ericsson at it worst and below the percent in may 2005. The data taken was $15860 * 10^6 = (15164 + 656) * 10^6$ stocks, where the latter sum indicates A and B shares. The face value of long-term debt was $97460MSEK$ in*

2002, and $33643 MSEK$ *in 2005. The annual volatiltiy calculated on a 60-day basis was set to* 0.3 *in may 2005 and* 1.4 *in 2002, the latter differing drastically from the 10-day volatility peak of 2.1. All stock was approximated to behave as B-stock and the values were approxiamted to* $3.6 SEK$ *in 2002 and* $23 SEK$ *in 2005. The credit asymmetry between face value and debt-holders value was assumed given by standard credit rating agencies as* 0.95.

The data above should be compared with the default probabilities given by credit agencies, such as such as Moody's and Standard and Poor. The Moodys credit rating of Ericsson was Ba2 in both 2002 and 2004. The Standard and Poor rating was BB+ at the end of 2004 and BB at the end of 2002.

4.1. **Physical versus Risk-neutral probabilities.** We have from a standard Girsanov transformation the following formula for the physical probability of default:

$$\begin{aligned} p_{physical} &= 1 - E^{t,x}(e^{-\int_t^T f(s,X_s)ds}\mathcal{D}(t,\omega))/E^{t,x}(\mathcal{D}(t,\omega)) \\ \mathcal{D}(t,\omega) &= e^{-\int_t^T \theta(s)^2/2ds + \int_t^T \theta(s)dB_s}, \\ \theta(t) &= -(\mu(t,X_t) - r(t,X_t))/\sigma(t,X_t) \\ p_{risk-neutral} &= 1 - E^{t,x}(e^{-\int_t^T f(s,X_s)ds}) \end{aligned}$$

5. A Multinomial Model of Stock-crash Occurences

In this section we present a model that redefines the concept of a stock-crash. The substitution is simple, we define for each company a implied risk-neutral stock crash intensity $f_i := f$ for $X_i := X_{company,i}$, and compute the physical default probability for each company. The number of stock crashes follows then a multinomial distribution. The partition function is given by

$$Z := \prod_1^n (p_i + q_i)$$

where

$$\begin{aligned} p_{physical,i} &= 1 - E^{t,x}(e^{-\int_t^T f_i(s,X_s^i)ds}\mathcal{D}(t,\omega))/E^{t,x}(\mathcal{D}(t,\omega)), \\ \mathcal{D}(t,\omega) &= e^{-\int_t^T \frac{\theta_i(s)}{2}^2 ds + \int_t^T \theta_i(s)dB_s} \\ \theta_i(t) &= -(\mu(t,X_t^i) - r(t,X_t^i))/\sigma_i(t,X_t^i) \\ p_{risk-neutral,i} &= 1 - E^{t,x}(e^{-\int_t^T f_i(s,X_s^i)ds}). \end{aligned}$$

Here either p_i is the risk-neutral or physical probability of default of the company i given above. The model above should be at its peak of accuracy when f_i is computed with finite maturity since this is a more realistic setting, but our model gives an ansatz for a better definition of default probability.

Example 5.1. *This time we investigate the risk-neutral default probability space of ABB, Ericsson and Alfa Laval at May 2005. From data taken from annual reports and stock exchange time series we obtain the following table:*

ERIC/ALFA	*Company survives*	*Company defaults*
Company survives	*0.9603*	*0.0097*
Company defaults	*0.0297*	*0.0003*

ERIC/ABB	*Company survives*	*Company defaults*
Company survives	*0.98505*	*0.00995*
Company defaults	*0.00495*	*0.000005*

ALFA/ABB	*Company survives*	*Company defaults*
Company survives	*0.96515*	*0.02985*
Company defaults	*0.00485*	*0.00015*

From these tables we see that diversification of credit risk is a good way to keep extreme events from happening too often. This of course assumes zero correlation for the moment.

6. A Multinomial Model of Stock-Crash Occurences Incorporating Correlation of Default

Defining the partition function(we now use multidimensional Brownian motion to model the stock market and ω is the path of the Brownian motion)

$$Z(t,\omega) := \prod_{1}^{n} (p_i(t,\omega) + q_i(t,\omega))$$

with $p_i(t,\omega) = 1 - e^{-\int_t^T f_i(s,X_s^i)ds}$, $q_i(t\omega) = 1 - p_i(t,\omega)$ we get the probability of the individual defaults from the integrated monomials in p_i and q_i the expansion

$$1 = Z_{t,x} = E(Z(t,\omega)\mathcal{D})/E(\mathcal{D}).$$

Since the expectation of the Doleans exponential is $E(\mathcal{D}) = 1$ by standard Ito calculus we see directly that the individual default event probabilities are given by

$$event = E(\epsilon_1(t,\omega)\epsilon_2(t,\omega)\cdots\epsilon_{n-1}(t,\omega)\epsilon_n(t_,\omega)\mathcal{D}).$$

where ϵ_i is either $p_i(t,\omega)$ or $q_i(t,\omega)$ depending on the event. Explicitly we have with $F_i = \int_t^T f_i(x_i,s)ds$ that $\partial_{x_i}F_i\sigma_i C^{i,j}\partial_{x_j}F_j\sigma_j$ is the infitisimal covariance between intensities, where $C^{i,j}$ is the covariance matrix of the Browninan motions, and this can be used to evaluate the terms in the partition function above. i.e this gives with $q_i = e^{-F_i}$ that:

$$q_i q_j = e^{-F_i} e^{-F_j} = 1 - F_i - F_j + \int_t^T \partial_{x_i} F_i \sigma_i C^{i,j} \partial_{x_j} F_j \sigma_j ds$$

In the above matrix of default intensities and probablities (Example 5.1) we have not included correlation, and the above formula or partition function is how to include this.

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