SUPERHEDGING, CLASSIFICATION AND MONOTONICITY OF CONTINGENT CLAIMS WITH LEVEL AND TIME DEPENDENT VOLATILITY, DRIFT AND RATE

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ABSTRACT. In this paper we show results on monotonicity and generalised greeks in rate, volatility and other parameter functions for options on both traded and non traded underlyings. This leads to a classification theory of options in terms of the contract function. We cover the cases of European, Barrier(knock-out), American, some Asian options as well as credit risky options. We also study options in incomplete markets. We characterize monotonicity in rate and volatility for European options up to logical equivalence as a condition on the contract function, something that yields a classification of options in call-like, put-like and options which are neither. Stochastic diffusions are assumed to drive the underlying through both time and space dependent drift and volatility under the martingale measure. Applications of these monotonicity results include superhedging and statistical arbitrage and the results are exhaustive in the Markovian setting on the sign of generalised greeks for the contingent claims mentioned.

1. Paper 1: Monotonicity of the Black-Scholes Functional

Since the discovery of the Black-Scholes equation numerous people have tried to prove various results concerning its properties. El Karoui et. al 1998 and Bergman et. al. 1996 showed that the Black-Scholes equation possesses a monotonicity in volatility function, thus generalizing the constant volatility of geometric Brownian motion to level and time dependent volatility by still having the same nice monotonicity properties as the standard Black-Scholes model which assumes constant rates and volatility. Bergman and others also showed a monotonicity property of European calls under upwards shifts in the term structure. These researchers seem to have been motivated in this pursuit by previous work by Merton(1973) and Jagannathan(1984) among other things showing that a call options price is an increasing, convex function of the current stock price. Cox & Ross(1976) showed that the price function of any European contingent claim inherits properties of the contract function, such as monotonicity.

In this paper we use the differential calculus of non-linear functionals as an alternative to the PDE approach. Both approaches are in other words taken below.

We are considering monotonicity properties since this is connected to hedging. If a option writer is mis-specifying the volatility by some positive function for hedging purposes and the value funtional of some contract is convex in stock space, it is then known that the issuing company will have overestimated what it will owe. Same kind of reasoning holds if it mis-specifies the rate. If the rate function that it uses is strictly bigger then it will have overestimated the value of an outstanding call and created a super-hedge if we have a call-like[1] contract. However, this has hithereto only been proved for deterministic rates and simple European call options.

The reasons for focusing on time and level dependent rates and volatilities are threefold. Firstly the case of deterministic rates has already been more or less exhausted. Secondly it is not inconceivable to model a rate as dependent on an underlying variable such as the debt over value ratio of a company or a stock index through a function. That the short rate of debt is dependent on the relative quantity of debt is due to matters such as default and counterparty risk in general and it is well known from elementary finance that there is a non-trivial interdependence between the evolution of stock markets and the behaviour of rates, although the nature of these dependencies is non-trivial. This is due e.g. to central bank incentive to stimulate an economy and lending as well as the amount of risk capital available to financial institutions, which has a positive correlation with positive evolution on stock markets. Thirdly, a common type of models in interest rate modelling are the so called affine ones, which typically result in rates that are functions of the underlying variables. The variables are called factors in interest rate theory. The special case with only one factor except for time is our situation.

As a bonus we obtain new proofs of old results as well as precise expressions for the difference between the value of an option for two different values in the infinite dimensional parameter space generated by volatilities and rates.

[1] This concept is a notion created by the author together with the concept put-like.

2. The Black-Scholes Model

We assume that the underlying stock or variable follows a one-dimensional stochastic diffusion under a martingale measure. Thus we have that

Hypothesis 2.1.

$$dX_t = (\mu(t, X_t) - \lambda\sigma(t, X_t))X_t dt + \sigma(t, X_t)X_t dB_t$$

where B_t is Brownian motion, μ is physical drift of X_t and σ is the volatility of X_t. λ is a continuous function called the market price of volatility risk. The relevant probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{t\leq s\leq T}, P)$ is as follows: It is the canonical Wiener space where Ω is the space of continuous functions $C([t,T], \mathbb{R})$, $(\mathcal{F}_s^0)$ is the filtration generated by the coordinate process $B_t(\omega)$, $\omega \in \Omega$, P is Wiener measure on $\mathcal{F}_T^0$, $\mathcal{F}$ is the P-completion of $\mathcal{F}_T^0$,and for each s, $\mathcal{F}_s$ is $\mathcal{F}_s^0$ completed with the null sets of $\mathcal{F}$.

Here we have also

Hypothesis 2.2.

- *$\mu(t, x) - \lambda(t, x)\sigma(t, x) \in \mathbb{R}$ and $\sigma(t, x) \geq 0$ are Lipschitz.*
- *σ is square integrable a.s when integrated versus the Lebesgue measure on the time interval $[0, T]$, Du and $\mu - \lambda\sigma$ is likewise integrable a.s when integrated versus Du. We drop this condition when considering credit risky options since we then explictly allow stock default.*
- *Both $(\mu(t, x) - \lambda(t, x)\sigma(t, x))x$ and $\sigma(t, x)x$ are at most asymptotically linear in growth in the spatial variable:*

$$|\sigma(t, x)| + |\mu - \lambda\sigma| \leq M(1 + \frac{1}{x}), M \in \mathbb{R}$$

- *The rate $r(t, x)$ is bounded and Lipschitz.*

These last conditions inforce existence of a strong solution to the SDE(See Karatzas et al(1988)., Definition 2.1 on page 285 and Theorem 2.9 on page 289). When the underlying is traded we also use $\mu(t, x) - \lambda(t, x)\sigma(t, x) \geq 0$. Thus for the case with traded underlyings the rate $r(t, x)$ is greater than zero, $r(t, x) \geq 0$.

The Black-Scholes equation is then

$$\begin{aligned}(\partial_t + \frac{\sigma(t, x)^2 x^2}{2}\partial_x^2 + (\mu(t, x) - \lambda\sigma(t, x))x\partial_x)F(t, x) = rF(t, x) \\ F(T, x) = g(x).\end{aligned}$$

on the domain $t \in [0, T], x \in \mathbb{R}_+$, see Duffie(1996), section F, Chapter 5, and Bjork(1998), Proposition 10.5. We also assume the following hypothesis:

Hypothesis 2.3. *Here g(x) is at most of polynomial growth and is continuous. The solution is required to be $C^{2,1}$ on the interior of the domain.*

Since for a traded underlying we have $\lambda = (\mu - r)/\sigma$ for the price of volatility risk, the Black-Scholes equation reduces to

$$\begin{aligned}(\partial_t + \frac{\sigma(t, x)^2 x^2}{2}\partial_x^2 + r(t, x)x\partial_x)F(t, x) = r(t, x)F(t, x) \\ F(T, x) = g(x).\end{aligned}$$

That is the case we shall first consider. Later we shall consider the case λ an independent parameter function, which arises in incomplete markets such as interest rate (IR) markets. Also, we have chosen to list our assumptions about credit risk derivatives later in the appropriate section since we then allow other features than above.

We shall use something called time ordered products to represent the Feynman-Kac solution to the Black-Scholes equation. The conditions that we have imposed on the SDE and the data $g(x)$ at $t = T$ guarantees the existence of such a solution, so this is relevant to our situation. We define the time ordered product to be the operator $\mathcal{T}$ that arranges products of operators that are at different times so that they are in time decreasing order from right to left. As some examples we thus have

Example 2.1. *Let $\mathcal{O}(t)$ be a family of operators indexed by t in some one-dimensional time continuum.Then*

$$\begin{aligned}&\mathcal{T}[\mathcal{O}(t = 3)\mathcal{O}(t = 5)\mathcal{O}(t = 2)] \\ &= \mathcal{O}(t = 2)\mathcal{O}(t = 3)\mathcal{O}(t = 5)\end{aligned}$$

We can represent the time ordering of two operators at different times by use of a Heaviside function θ.

$$\mathcal{T}[\mathcal{O}(t)\mathcal{O}(t')]$$
$$= \mathcal{O}(t)\mathcal{O}(t')\theta(t'-t) + \mathcal{O}(t')\mathcal{O}(t)\theta(t-t')$$

This has natural generalizations in terms of characteristic functions of hypertriangles for products of higher degrees. For operators at equal times the time ordering is defined by the normalized totally symmetric function of the operators. We give a further example that is more close to the applications we have in mind.

Definition 2.1. *We define*

$$\mathcal{L} = \frac{\sigma(t,x)^2 x^2}{2}\partial_x^2 + (\mu - \lambda\sigma)x\partial_x - r(t,x)$$

This definition holds throughout the paper. Note that for the case of traded underlyings this reduces to

$$\mathcal{L} = \frac{\sigma(t,x)^2 x^2}{2}\partial_x^2 + r(t,x)x\partial_x - r(t,x)$$

Example 2.2.

$$\mathcal{T}[e^{\int_t^T \mathcal{L}(t)dt}] = \mathcal{T}[\sum_0^\infty \frac{(\int_t^T \mathcal{L}(t)dt)^n}{n!}]$$
$$= \sum_0^\infty \frac{\int_t^T \cdots \int_t^T \mathcal{T}[\mathcal{L}(t_1)\mathcal{L}(t_2)\cdots\mathcal{L}(t_n)]dt_1 dt_2 \cdots dt_n}{n!}$$

Since the space of equal times in each multidimensional integration is a Lebesgue null set we also note that under some regularity assumptions we need not bother about time ordering at equal times for expressions such as the above.We give an example of the use of time ordering operators:

Theorem 2.1. *A solution to the Black-Scholes equation with C^∞ last value data and null boundary conditions is then given by*

$$F(t,x) = \mathcal{T}[e^{\int_t^T \mathcal{L}(t)dt}]F(T,x) = \mathcal{T}[\sum_0^\infty \frac{(\int_t^T \mathcal{L}(t)dt)^n}{n!}]g(x)$$

For continuous last value data we define the solution by successive approximation in the domain of the partial differential operator by C^∞ functions.

Proof. We have

$$\partial_t \mathcal{T}[e^{\int_t^T \mathcal{L}(t)dt}]F(T,x)$$
$$= \partial_t \mathcal{T}[\sum_0^\infty \frac{(\int_t^T \mathcal{L}(t)dt)^n}{n!}]g(x)$$
$$= \mathcal{T}[-\mathcal{L}(t)e^{\int_t^T \mathcal{L}(t)dt}]g(x)$$
$$= -\mathcal{L}(t)\mathcal{T}[e^{\int_t^T \mathcal{L}(t)dt}]g(x)$$

We note also that the expression above satisfies the appropriate terminal data. Hence the expression satisfies the equation. When the last value data is only continuous the successive approximation procedure defines a unique solution by using the Hadamard property of the PDE(see the appendix for the Hadamard property).

In fact the expression above is the operator representation of the Feynman-Kac solution.See Peskin et al. (1997) □

In the following we define δ as the exterior (Frechet) differential on the Hilbert space $L^2([0,T] \times \Omega, dt \times dx)$ viewed as a Hilbert manifold of trivial homotopy type- i.e a contractible infinite dimensional manifold. For the calculus of variation, something that we will lean on later, see the two volume treatise on physical mathematics by Yvonne Choquet-Bruhat and Cecile DeWitt-Morette called Analysis, Manifolds and Physics(1999). The chapter differential calculus on Banach spaces, pages 71-109,is the most relevant to our discussion, in particular the (standard) definition on page 71 of the Frechet derivative. The material on Banach manifolds is also interesting and is addressed in pages 504-601. We assume the reader to be familiar with the basics of such analysis, at least in the finite dimensional case.

Definition 2.2. *A Banach Manifold is a topological space locally homemorphic to a Banach space, i.e locally homemorphic to a complete normed space.*

Definition 2.3. *The Frechet differential of a functional $F(\rho)$ at ρ on a Banach space X is defined as the linear functional DF satisfying*

$$F(\rho + h) - F(\rho) = DF(h) + o(h)$$

where $\rho \in X$, $h \in X$. Here $o(h) = R(h)||h||_X$ where R is a bounded functional in a neighbourhood of the origin and $|| \cdot ||_X$ is the norm on X.

The exterior Frechet differential is defined in any local chart defined by a local homemorphism as the Frechet differential $\sum_x d\rho^x \wedge DF_x$ acting on sections of the exterior cotangent bundle of the Banach manifold in question. Here $\wedge$ is the wedge or exterior product defined by total antisymmetrization of tensors.We also use $\wedge$ as the minimum of two quantities later, but the contexts are quite different so this should not cause any confusion.

3. The Main Theorem

We are now ready to prove

Theorem 3.1 (Main Theorem)**.** *Assume that the Black-Scholes functional is continuously Frechet differentiable and that we have null boundary conditions. We have that ΔF is either positive semidefinite or negative semidefinite,i.e*

$$\int_\gamma \delta F = \int_{\rho_0=r_0\times\mu_0\times\sigma_0\times\lambda_0}^{\rho_1=r_1\times\mu_1\times\sigma_1\times\lambda_1} \delta F = F(\rho_1) - F(\rho_0) = \Delta F \geq 0$$

or

$$\int_\gamma \delta F = \int_{\rho_0=r_0\times\mu_0\times\sigma_0\times\lambda_0}^{\rho_1=r_1\times\mu_1\times\sigma_1\times\lambda_1} \delta F = F(\rho_1) - F(\rho_0) = \Delta F \leq 0$$

for all ρ_1 in the path $\gamma =< \rho_0, \rho_2 >$, where ${\rho_1 \geq \rho_0}^2$ if and only if the sign of

$$\frac{\partial \int_t^T \mathcal{L}}{\partial \rho^s(x)} F(s,x,\rho)|_{\rho\in\gamma} = \frac{\partial \mathcal{L}(s)}{\partial \rho^s} F(s,x,\rho)|_{\rho\in\gamma}$$

is of one type-either positive semidefinite or negative semidefinite respectively. Here $\rho^s(x) = \rho(s,x)$, $s \in [t,T]$. We use $\rho^s = \rho^s(x)$ as coordinates on our Banach manifold $L^2([0,T]\times\mathbb{R}_+, ds\times dx)$, so ρ_1 and ρ_0 only differ by a square integrable perturbation.

Proof. We use the operator representation of the Feynman-Kac solution. We have

$$\begin{aligned}\delta F(t,x) &= \delta\mathcal{T}[e^{\int_t^T \mathcal{L}}]F(T,x) \\ &= \mathcal{T}[(\delta\int_t^T \mathcal{L})e^{\int_t^T \mathcal{L}}]F(T,x) \\ &= \mathcal{T}[(\int_t^T \frac{\partial \mathcal{L}(s)}{\partial \rho^s}\delta\rho^s ds)e^{\int_t^T \mathcal{L}}]F(T,x) \\ &= \int_t^T \mathcal{T}[(\frac{\partial \mathcal{L}(s)}{\partial \rho^s})e^{\int_t^T \mathcal{L}}]F(T,x)\delta\rho^s ds\end{aligned}$$

In the above we used the composite mapping theorem for Frechet derivatives, see page 73, Choquet-Bruhat et al(1971), as well as differentiation under the integral sign, which we can do since we have a finite measure space and the integrand is differentiable. Considering the component partial (Frechet) derivatives we have

$$\begin{aligned}\frac{\partial F(t,x)}{\partial \rho^s} &= \\ &= \mathcal{T}[(\frac{\partial \mathcal{L}(s)}{\partial \rho^s})e^{\int_t^T \mathcal{L}}]F(T,x) \\ &= \mathcal{T}[e^{\int_t^s \mathcal{L}}\frac{\partial \mathcal{L}(s)}{\partial \rho^s}e^{\int_s^T \mathcal{L}}]F(T,x) \\ &= \mathcal{T}[e^{\int_t^s \mathcal{L}}]\frac{\partial \mathcal{L}(s)}{\partial \rho^s}F(s,x)\end{aligned}$$

This last expression is positive semi definite or negative definite at all t if and only if

$$\frac{\partial \mathcal{L}(s)}{\partial \rho^s}F(s,x) \geq 0$$

or

$$\frac{\partial \mathcal{L}(s)}{\partial \rho^s}F(s,x) \leq 0$$

since the Black-Scholes equation preserves signs of contracts. That this preservation of signs we now used occurs for parabolic PDE of second order is obvious in view of the stochastic representation of the solution. Hence going to bigger $\rho^s = \rho(s,x)$ we have

$$\int_\gamma \delta F = \int_\gamma \int_t^T \mathcal{T}[(e^{\int_t^s \mathcal{L}}]\frac{\partial \mathcal{L}(s)}{\partial \rho^s})F(s,x)d\rho^s ds \geq 0$$

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[2]Here we define $\rho_1 \geq \rho_0$ when the component functions in the cartesian product $\rho_1 = r_1\times\mu_1\times\sigma_1\times\lambda_1$ dominate the component functions in $\rho_0 = r_0\times\mu_0\times\sigma_0\times\lambda_0$, i.e.if and only if $r_1 \geq r0, \mu_1 \geq \mu_0, \sigma_1 \geq \sigma_0, \lambda_1 \geq \lambda_0$.

or

$$\int_\gamma \delta F = \int_\gamma \int_t^T \mathcal{T}[(e^{\int_t^s \mathcal{L}}]\frac{\partial \mathcal{L}(s)}{\partial \rho^s})F(s,x)d\rho^s ds \leq 0$$

We also have the converse, since definiteness of continuous integrand on an arbitrary domain is equivalent to definiteness of the integral over arbitrary domains(That the integrand is continuous follows by continuous Frechet differentiability.). □

Let ρ be as above a parametrization of a path in rate, drift, market price of risk and volatility space going in increasing direction,(i.e going away from the origin if the path is embeddable in L^2 or in the general case just e.g. going to higher volatility or rate).

Theorem 3.2 (Main theorem II, PDE version). *Assume that $F(t,x,\rho)$ satisfies the Black-Scholes equation*

$$(\partial_t + \mathcal{L}(\rho))F(t,x,\rho) = 0$$
$$F(T,x,\rho) = g(x).$$

for different parameter values ρ, and that the associated operator $\mathcal{L}$ is partial differentiable in ρ. Assume furthermore Hypothesis 2.3 and either 1) Hypothesis 2.2 or that 2) the pertaining SDE diffusion is absorbing at $x = 0$ as well as that the contract function is vanishing at the boundary. We have, going in the direction of increasing parameter functions,

$$\Delta F = F(\rho_2) - F(\rho_1) \geq 0$$

or

$$\Delta F = F(\rho_2) - F(\rho_1) \leq 0$$

if and only if

$$\partial_\rho \mathcal{L} F \geq 0$$

or

$$\partial_\rho \mathcal{L} F \leq 0$$

respectively. Here the derivative ∂_ρ is a usual partial derivative.

Proof. Subtracting one of the Black-Scholes equations

$$(\partial_t + \mathcal{L}(\rho_1)F(t,x,\rho_1) = 0$$
$$F(T,x,\rho_1) = g(x).$$

from the other Black-Scholes equation

$$(\partial_t + \mathcal{L}(\rho_2))F(t,x,\rho_2) = 0$$
$$F(T,x,\rho_2) = g(x).$$

we obtain for products from $\Delta FG = (\Delta F)G(\rho_2) + F(\rho_1)\Delta G$ and linearity

$$0 = (\partial_t + \mathcal{L}(\rho_1))\Delta F + \Delta \mathcal{L} F(\rho_2)$$

Since ΔF satisfies null last value data (the boundary value data are the difference between the boundary value data for the two different problems) the solution is given by a Feynman-Kac representation (See Karatzas and Shreve(1991), Theorem 7.6. for a version on $\mathbb{R}^n$.) as

$$\Delta F = E\int_t^T e^{-\int_t^s r(u,X_u)du}\Delta \mathcal{L} F(t,X_s,\rho_2)ds,$$

This is seen since the term

$$\int_t^T p(t,x,s=\tau_D)e^{-\int_t^s r(u,x=0)}\Delta F(s,0)ds$$

vanishes when either the boundary is almost surely not reached, which is among other things implied by Hypothesis 2.2, or when the contract function is zero at the boundary and the boundary is absorbing.(Here τ_D is the stopping time for reaching the boundary at $x = 0$.) This is so in the first case since

$$X_{t,x} = xe^{\int_0^t (r-\frac{\sigma^2}{2})ds + \int_0^t \sigma dB_s}$$

and the conditions of a.s integrability in Hypothesis 2.2 mean that the exponential is finite almost surely, hence $X \neq 0$ almost surely. The second case with absorbing boundary means that $\Delta F = 0$ since $\Delta g(x=0) = 0$. Thus in view of this Feynman-Kac representation we conclude sufficiency by linearity. In as far as necessity note $(\Delta \mathcal{L})F = (\Delta\rho\frac{\partial \mathcal{L}}{\partial \rho})F + O((\Delta\rho)^2)$. It is crucial to note that the error term $O((\Delta\rho)^2)$, denoting a multiple of a bounded function in ρ-space with the obvious monomial $(\Delta\rho)^2$, can be neglected by the Hadamard property. If we thus choose a very small $\Delta\rho$ that has $supp\Delta\rho \subset OSS$ where OSS denotes the set where OSS denotes the opposite sign set of $\partial\rho\mathcal{L}F$ we obtain a ΔF of the opposite sign by linearity of the Feynman-Kac representation. This can happen only if OSS is not the empty set, hence we are done. □

Example 3.1. *Set $\rho^s = \sigma(s,x)$,so that we are fixing the remaining infinite dimensional parameters and are going in path γ of increasing volatilities. Assume we are dealing with a traded underlying x. Then*

$$(\partial_{\rho^s}\mathcal{L}(s,x))F(s,x) = \sigma x^2\partial_x^2 F(s,x)$$

which is positive semidefinite if and only if $\partial_x^2 F \geq 0$, i.e that the contract is convex and that pertaining equation preserves convexity. Here we implicitly assumed the contract to be C^2. By Hadamard well posedness- of the Black-Scholes equation we have that approximating a convex contract function by C^∞ functions we will approximate the solution arbitrarily well. Hence this is true for arbitrary convex contract functions preserved in convexity by the Black-Scholes equation. See the appendices for proof of Hadamard well-posedness of the Black-Scholes equation.

Thus in view of the above example we have proved the following theorem

Theorem 3.3. *Assume that we are given the Black-Scholes equation for a traded underlying asset and level and time dependent rates and volatilities. Then if the Black Scholes functional is continuously Frechet diffrentiable we have that it is increasing in volatility if and only if the contract function is convex and the Black-Scholes equation preserves convexity.*

Thus we directly note the corollary

Corollary 3.1. *Assume that we are given the Black-Scholes equation for a traded underlying asset and level and time dependent volatilities and only time dependent rates. Then if the Black Scholes functional is continuously Frechet differentiable we have that it is increasing in volatility if and only if the contract function is convex.*

Proof. That the Black-Scholes equation preserves convexity for deterministic rates is well known. □

Our proof of the corollary above is simpler than the usual. We do not give an example of this corollary since it is well known to hold, see e.g. Bergman et al.(1996).

Assumption 3.1. *We assume henceforth whenever we use Main Theorem I that all Black-Scholes functionals we consider are continuously Frechet differentiable so that it applies. As an alternative when we use Main Theorem II we may assume that the differential operator generating the Black-Scholes equation is differentiable w.r.t ρ.*

Assumption 3.2. *We shall also always assume that we are dealing with Lebesgue square integrable perturbations of the parameter functions when we use Main Theorem I. Our monotonicity results thus only apply when the parameter functions differ by such a (positive semidefinite) Lebesgue square integrable function. However, when we use Main Theorem II this is not a necessary assumption.*

4. Applications of the Main Theorem

We begin with a lemma.

Lemma 4.1. *Assume that we are studying an option on a traded asset with time and level dependent rate $r = r(t,x)$ and volatility $\sigma = \sigma(t,x)$ and continuous contract function g. Assume sufficient conditions for Hadamard continuity as given in the appendix. Assume also Hypothesis 2.1, 2.2 and 2.3. Then we also have*

$$(x\partial_x - 1)F \leq 0$$

is equivalent to

$$(x\partial_x - 1)g \leq 0$$

meant in the sense of distributions for the general case of a continuous contract. The other scenario also holds, again in the sense of distributions;

$$(x\partial_x - 1)F \geq 0$$

is equivalent to

$$(x\partial_x - 1)g \geq 0$$

Proof. Assume first smooth coefficients in the Black-scholes PDE. Locally we have for the Black-Scholes functional that the sign of the expression $(x\partial_x - 1)g$ is conserved if and only if $(x\partial_x - 1)\mathcal{L}g$ is either positive or negative semidefinite. By using standard commutation relations for algebras of differential operators, we get

$$\begin{aligned}(x\partial_x - 1)\mathcal{L}g &= [x\partial_x - 1, \mathcal{L}]g + \mathcal{L}(x\partial_x - 1)g \\ &= \frac{x\partial_x\sigma^2}{2}x^2\partial_x^2 g + (x\partial_x r)(x\partial_x - 1)g + \mathcal{L}(x\partial_x - 1)g.\end{aligned}$$

However, noting $x^2\partial_x^2 = x\partial_x(x\partial_x - 1)$ we get

$$\begin{aligned}(x\partial_x - 1)\mathcal{L}g &= [x\partial_x - 1, \mathcal{L}]g + \mathcal{L}(x\partial_x - 1)g \\ &= (\frac{x\partial_x\sigma^2}{2}x\partial_x + x\partial_x r)(x\partial_x - 1)g + \mathcal{L}(x\partial_x - 1)g \\ &= (\frac{x\partial_x\sigma^2}{2}x\partial_x + (x\partial_x r) + \mathcal{L})(x\partial_x - 1)g \\ &= \mathcal{K}(x\partial_x - 1)g\end{aligned}$$

Since $\mathcal{K}$ is obviously an elliptic operator the diffusion it generates preserves signs, hence we are done proving sufficiency. By the Hadamard property we can approximate the solution for continuous coefficients by solutions generated by approximating smooth coefficients, hence for an approximating sequence of smoothe coefficients we get $\lim F_n = F$ uniformly and $(x\partial_x - 1)F_n \geq 0$ or $(x\partial_x - 1)F_n \leq 0$ holds, thus we are done since either $\lim(x\partial_x - 1)F_n \geq 0$ or $\lim(x\partial_x - 1)F_n \leq 0$, were the limit is pointwise. Necessity on the other hand is obvious in view of that g is the restriction of F to time $t = T$. □

Example 4.1. *A call with strike K satisfies $(x\partial_x - 1)g = K\theta(x - K) \geq 0$ where θ is the Heaviside function. A Put on the other hand satisfies $(x\partial_x - 1)g = -K\theta(K - x) \leq 0$. Puts and calls are thus belonging to the two different cases mentioned above.*

We also have the lemma

Lemma 4.2. *Assume the Black-Scholes equation for a traded underlying and general time and level dependent rates and volatilities. Assume furthermore Hypothesis 2.1,2.2 and 2.3. If $(x\partial_x - 1)F$ is either positive semidefinite or negative semidefinite, then and only then is it increasing or decreasing in all rates respectively.*

Proof. Since we have $\partial_\rho\mathcal{L} = \partial_r\mathcal{L} = x\partial_x - 1$ we are done by using the Main Theorem in either version.

□

Theorem 4.1 (Main Theorem on Rate Monotonicity)**.** *Assume time and level dependent volatilities and rates and contiuous contract functions as well as assuming Hypothesis 2.1,2.2 and 2.3. We have that $(x\partial_x - 1)g \leq 0$ in the sense of distributions is necessary and sufficient for an option on a traded asset to be decreasing in rates. Likewise $(x\partial_x - 1)g \geq 0$ in the sense of distributions is necessary and sufficient for an option on a traded asset to be increasing in rates.*

Proof. By Lemma 4.1, which states that $(x\partial_x - 1)g \leq 0$ is equivalent with $(x\partial_x - 1)F \leq 0$ under such circumstances, and Lemma 4.2 this is imminent. □

Remark 4.1. *The assertions of the theorem above are also true for the case with absorbing boundary and contract functions satisfying $(x\partial_x - 1)g \geq 0$, since then the contract is vanishing at the origin then. For example, any bounded volatilities and rates would yield such a case.*

We can reformulate the theorem for those who do not like to use distributions but are willing to disallow the origin in stock space, i.e disallow default.

Example 4.2. *Geometric brownian motion does not default almost surely. Neither do CEV(Constant Elastisity of Variance) models described by*

$$dX_t = rX_t dt + \sigma X_t^{\alpha} dB_t$$

with r and σ constant for $\alpha > 1/2$.

Theorem 4.2 (Main Theorem on Rate Monotonicity)**.** *Assume time and level dependent volatilities and rates and continuous contract functions and Hypothesis 2.1,2.2 and 2.3. We have that $\frac{g}{x}$ decreasing is necessary and sufficient for an option on a traded asset to be decreasing in rates. Also $\frac{g}{x}$ increasing is necessary and sufficient for an option on a traded asset to be increasing in rates.*

Proof. Since we have that the above conditions on $\frac{g}{x}$ are equivalent to either $(x\partial_x - 1)g \leq 0$ or $(x\partial_x - 1)g \geq 0$ when the contract function is smooth we are done by using a successive smooth uniform approximation of the contract function satisfying the appropriate inequality. □

Remark 4.2. *The conclusion of the above theorem actually also holds under deterministic dividends. This is seen by inspection of the proofs.*

Remark 4.3. *The property $(x\partial_x - 1)g \geq 0$ can be described by saying that the area under the graph of the contract function always lies under a straight line from the origin ending on the graph. Likewise, the case with $(x\partial_x - 1)g \leq 0$ can be described by saying that a line from the origin ending on the graph is always under the graph.*

Definition 4.1. *We call a option with a contract satisfying $(x\partial_x - 1)g \geq 0$ a call-like option. Conversely $(x\partial_x - 1)g \leq 0$ defines a put-like option.*

Remark 4.4 (The General Call and Put)**.** *Recall that a call satisfies $(x\partial_x - 1)g = K\theta(x-K) \geq 0$. Thus a call is increasing in rates under our general assumptions about the form of time and level dependent rate and volatility. Likewise the put is decreasing in rate by $(x\partial_x - 1)g = -K\theta(K-x) \leq 0$. Since the condition $(x\partial_x - 1)g \geq 0$ generalizes the call and similarly for the put this motivates the definition above. The theorem above thus says that a option is increasing in rate if and only if it is a call-like option and decreasing in rate if and only if it is a put-like option.*

Example 4.3. *Consider a bond $B(t,T,x) = F(t,x)$ for some deterministic constant rate r . Then*

$$\partial_r F = -(T-t)B(t,T) \leq 0$$

Hence we have a decreasing property in rate, just as required.

Example 4.4 (The Geometric Brownian Motion Put, Explicitly)**.** *Consider the standard Black-Scholes model with geometric Brownian motion. Then for a standard European put F with strike K we have, using the well known formula for the value of a call and put-call parity,*

$$\partial_r F = (T-t)Ke^{-r(T-t)}(N(d_2) - 1) \leq 0$$
$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\{/ln(x/K) + (r - \frac{\sigma^2}{2})(T-t)\}.$$

Hence a put is decreasing in constant shifts of the rate, as should be by our theorem. Our theorem applies to arbitrary non-constant upward shifts of the rate, so this is just a very specific case.

Example 4.5 (The Geometric Brownian Motion Call, Explicitly). *Consider the standard Black-Scholes model with geometric Brownian motion. Then for a standard European call $C(t,x) = F(t,x)$ with strike K we have*

$$\partial_r C(t,x) = (T-t)Ke^{-r(T-t)}N(d_2) \geq 0$$
$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\{ln(x/K) + (r - \frac{\sigma^2}{2})(T-t)\}.$$

Hence a call is increasing in constant shifts of the rate, as should be by our theorem above.

Theorem 4.3. *Assume we are given a European option with convex and decreasing contract function for integrable deterministic rates $r(t)$. Assume that we consider a non-traded underlying following a diffusion under some martingale measure given by an exogenously defined volatility market price of risk $\lambda(t,x) \geq 0$ and that the drift under the martingale measure is an affine function, $(\mu - \sigma\lambda)x = a(t) + b(t)x$. Then the Black-Scholes functional is increasing in volatility.*

Proof. Computing the relevant derivative we have for decreasing smooth approximations of the contract function

$$\partial_{\rho^s}\mathcal{L}(s)F = \partial_{\sigma^s}\mathcal{L}(s)F = (\sigma x^2\partial_x^2 - x\lambda\partial_x)F \geq 0.$$

Here we used that convexity of the contract function is inherited by F since the rate is deterministic and the drift is affine, see S. Jansson and J. Tysk (2004). By Hadamard well-posedness, we are done. □

Example 4.6. *Consider a temperature modeled by geometric Brownian motion. Then for constant rates we have for a put P on the temperature $g(x) = (K-x)^+$ the following price in temperature(P should be multiplied by a constant having units dollars per temperature to get the usual price):*

$$C(t,x) = xe^{(\mu-\lambda\sigma-r)(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$C(t,X) - P(t,x) = xe^{(\mu-\lambda\sigma-r)(T-t)} - Ke^{-r(T-t)}$$
$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\{ln(x/K) + (\mu - \lambda\sigma + \frac{\sigma^2}{2})(T-t)\}, d_2 = d_1 - \sigma\sqrt{T-t}$$

Plotting this P in the volatility parameter we get the expected behaviour.

Theorem 4.4. *Assume that we are given a European option on a non-traded asset. If the contract function g is decreasing and the rate is increasing in x, then the option is increasing in the market price of risk $\lambda(t,x)$. On the other hand if the contract is increasing and the rate is decreasing, then the option is decreasing in $\lambda(t,x)$.*

Proof. We have, using the same reasoning as in the proof of Theorem 4.1, monotonicity of the option by the conditions on rate and contract functions fixed above. The option is either decreasing in the first case or increasing in the second case. Since for either increasing or decreasing smooth approximations of the contract function we have

$$\partial_{\rho^s}\mathcal{L}(s)F = \partial_{\lambda^s}\mathcal{L}(s)F = -x\sigma\partial_x F \geq 0$$

or

$$\partial_{\rho^s}\mathcal{L}(s)F = \partial_{\lambda^s}\mathcal{L}(s)F = -x\sigma\partial_x F \leq 0$$

respectively. We are now done by using the main theorem and Hadamard well-posedness. □

Example 4.7. *For constant rates we have for the temperature put option in example* 4.1*, plotting the λ behaviour, what we expect. The graph is increasing.*

Theorem 4.5. *Assume that we are given a European option on a non-traded variable. If the contract function g is decreasing and the rate is increasing in x, then the option is decreasing in the physical drift $\mu(t,x)$. On the other hand if the contract is increasing and the rate is decreasing, then the option is increasing in $\mu(t,x)$.*

Proof. We have, using the same reasoning as in the proof of Theorem 4.1, monotonicity of the option by the conditions on rate and contract functions fixed above. The option is either decreasing in the first case or increasing in the second case. Since for either increasing or decreasing smooth approximations of the contract function we have

$$\partial_{\rho^s}\mathcal{L}(s)F = \partial_{\mu^s}\mathcal{L}(s)F = x\partial_x F \leq 0$$

or

$$\partial_{\rho^s}\mathcal{L}(s)F = \partial_{\mu^s}\mathcal{L}(s)F = x\partial_x F \geq 0$$

respectively. We are done by using the main theorem and Hadamard well-posedness. □

Example 4.8. *Plotting the temperature put above for increasing drifts we have a decreasing behaviour in the value of the option. This is consistent with our theorem.*

We can also investigate monotonicity properties under changes in volatility for options on traded assets. Here is a theorem that attempts this.

Lemma 4.3. *Assume that we are dealing with an option on a traded asset, smooth contracts and Hypothesis 2.1,2.2 and 2.3. Assume furthermore that sufficient conditions for the Hadamard property to hold as mentioned in the appendix. Assume that either the rate is spatially convex and $(x\partial_x - 1)g \geq 0$ or that the rate is spatially concave and $(x\partial_x - 1)g \leq 0$. Then convexity of the option price is equivalent to convexity of the contract.*

Proof. We approximate the contract function with a smooth convex g and smooth coefficients and try to prove sufficiency. We have using the same method of proof as in Lemma 4.1 that

$$\begin{aligned}
x^2\partial_x^2(F(t-\Delta t) - F(t)) = \\
&\Delta t x^2\partial_x^2\mathcal{L}F(t) = \Delta t[x^2\partial_x^2, \mathcal{L}]F(t) + \Delta t\mathcal{L}(x^2\partial_x^2 F(t)) \\
&= \Delta t((x\partial_x\sigma^2 x\partial_x + \frac{1}{2}x^2\partial_x^2\sigma^2 + 2x\partial_x r)x^2\partial_x^2 F(t) \\
&+ x^2\partial_x^2 r((x\partial_x - 1)F(t)) + \Delta t\mathcal{L}(x^2\partial_x^2)F(t) \\
&= \Delta t\mathcal{K}(x^2\partial_x^2)F(t) + \Delta t(x^2\partial_x^2 r)((x\partial_x - 1)F(t))
\end{aligned}$$

for the appropriate expression valued at time $t-\Delta t$, Δt small and non-negative. The operator $\mathcal{K}$ is second order elliptic, hence it preserves signs since it generates a diffusion which preserves signs. The other term is positive by assumption by Lemma 4.1, hence we are done proving sufficiency in the general case for continuous coefficients by using the Hadamard property under approximation of coefficients. Necessity is obvious in view of the fact that g is the restriction of F to $t = T$, thus we are done. □

In view of the above theorem, we have the following corollary immediately.

Theorem 4.6. *A European option with smooth contract on a traded asset satisfying Hypothesis 2.1,2.2 and 2.3 is monotone in rate and increasing in volatility if and only if either $(x\partial_x - 1)g \geq 0$, g is convex and the rate is convex in the spatial variable or $(x\partial_x - 1)g \leq 0$, g is convex and the rate is concave in the spatial variable. In the first case it is increasing in rate and the latter it is decreasing in rate.*

Proof. In view of Theorem 3.3 and Lemma 4.3 we are done proving in one direction. In as far as the other direction, take a contract satisfying $(x\partial_x - 1)g = 1$ then choosing a non-convex rate we have a contradiction by the expression for preservation of convexity in the previous proof. For the other case, take a contract satisfying $(x\partial_x - 1)g = -1$ and a non-concave rate-this leads to a new contradiction. □

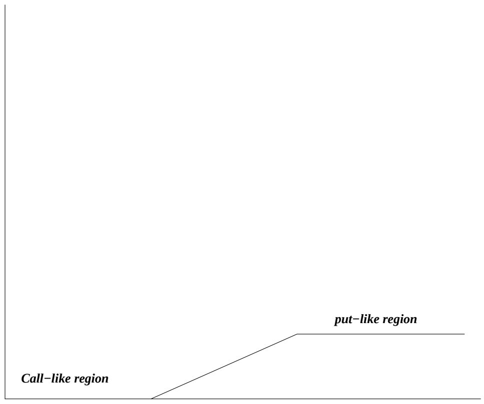

FIGURE 1. The contract in the example above. The horisontal axis is stock price while the vertical is value of contract function. In the call-like option region it is increasing in rate and in the put-like option region it is decreasing in rate. Incidentally, in roughly the two same regions for this contract we have two different monotonicity properties in volatility, increasing in the convex region and decreasing in the concave region.

Remark 4.5. *In the above two theorems we can generalize directly to the case with only continuous contract functions by an appropriate limiting procedure of contract functions using the Hadamard property of the Black-Scholes equation. The appropriate conditions are then for the contract functions $g(x)/x$ increasing and $g(x)/x$ decreasing respectively. The conditions on rates pertaining to the two different cases remain the same. The assertions remain true for deterministic dividends as is seen by inspection of the proof.*

Remark 4.6. *The assertions of the theorem above are also true for the case with absorbing boundary and contract functions satisfying $(x\partial_x - 1)g \geq 0$, since then the contract is vanishing at the origin then. For example, any bounded volatilities and rates would yield such a case.*

Example 4.9. *The contracts that we defined to be call-like in Definition 4.1 are if they posess convex contract function also increasing in volatility if and only if the rate is spatially convex in stock space. This is a generalization to level dependent rates of the usual properties of calls. Likewise a put-like option with convex contract is increasing in volatility if and only if the rate is concave.*

Example 4.10. *Assume we have an option a, say, smooth contract function g satisfying $(x\partial_x - 1)g \geq 0$ in $D \subset \mathbb{R}_+ \cup \{0\}$ and $(x\partial_x - 1)g \leq 0$ in $D^c \subset \mathbb{R}_+ \cup \{0\}$. Then also $(x\partial_x - 1)F \geq 0$ on $C \subset (\mathbb{R}_+ \cup \{0\}) \times [0,T]$ and $(x\partial_x - 1)F \leq 0$ on $C^c \subset (\mathbb{R}_+ \cup \{0\}) \times [0,T]$. Thus*

$$\Delta F = \int_t^T E(e^{\int_t^u r(s,X_s)ds} \Delta r(x\partial_x - 1)F)du$$

This is greater than zero if Δr has support in C and less than zero if Δr has support in C^c. An example of such a contract function is e.g. the difference between two calls $g = (x - K_1)^+ - (x - K_2)^+$, with $K_2 > K_1$, since then $(x\partial_x - 1)g = K_1\theta(x - K - 1) - K_2\theta(x - K_2)$ assumes both signs, and consequently the option is not monotone in rate. The same kind of argument shows that this portfolio is not monotone in volatility either, since the contract is neither convex nor concave.

Example 4.11 (Credit Risk Migration)**.** *We note that if a European option option is issued by a company and this company tries to value this option,then the rate it should use is the rate of its debt. This in turn would be directly dependent on the value over debt ratio of the company. Thus*

$$r = r(\frac{\theta x}{D(t)}, t)$$

where we can set the debt $D(t)$ to be a deterministic function. The deterministic function θ here is the amount of stocks issued by the company, and we only assume one type of stock to be present. If we are dealing with a call, the rate function will be decreasing as we go to infinity in stock space on financial grounds(an abundance of money in the market means that rates must be low in order for some people to borrow at all) as well as taking into account the theorem on monotonicity in volatility. Thus as the credit of the company changes,the discounted value of the option contract decreases since there is a suppressing factor in the in the form of the exponential $e^{-\int_t^T rds}$ under the expectation in the Feynman-Kac formula for the option. Hence this models credit migration.

Example 4.12 (Knock-out Barrier options)**.** *Say we want to investigate a put-like or call-like option that is knocked out in some rectangular domain D in the space-time of the Black-Scholes PDE. By our monotonicity results we see that this is given by formally letting $r(t,x) \to \infty$ in D in the put-like case and $r(t,x) \to -\infty$ in D in the call-like case. In practice this means having sufficiently large bounds on the rate. Alternatively we can impose null boundary conditions and restrict the domain of the Black-Scholes equation to the continuation region. This ideas above are a generalization of Baaquie's method for studying barrier options in the references. We then directly obtain the following theorems:*

Theorem 4.7 (Rate Monotonicity, Barrier options)**.** *Assume time and level dependent volatilities and rates and contiuous contract functions as well as assuming Hypothesis 2.1,2.2 and 2.3. We have that $(x\partial_x - 1)g \leq 0$ in the sense of distributions is necessary and sufficient for a barrier knock-out option to be decreasing in rates. Likewise $(x\partial_x - 1)g \geq 0$ in the sense of distributions is necessary and sufficient for an this option on a traded asset to be increasing in rates.*

Theorem 4.8 (Volatility Monotonicity, Barrier options)**.** *A European Barrier knock-out option with smooth contract on a traded asset satisfying Hypothesis 2.1,2.2 and 2.3 is monotone in rate and increasing in volatility if and only if either $(\partial_x - 1)g \geq 0$, g is convex and the rate is convex in the spatial variable or $(x\partial_x - 1)g \leq 0$, g is convex and the rate is concave in the spatial variable. In the first case it is increasing in rate and the latter it is decreasing in rate.*

Proof. The proofs are completely analagous to the previos proofs. Since the knock-out boundary inforces a restriction of the Black-Scholes equation domain with null boundary value data at the boundary we are done since then Main Theorem II holds under this new domain for the Black-Scholes operator. □

The possibly simplest other option to analyze is probably a 'mean' contract. It is somewhat interesting because it is as an exotic contract that may shed light on a special case of Asian options. By the above results we get these two theorems:

Corollary 4.1. *Assume that we have a 'Asian' or 'mean' contract on a traded asset $g = \frac{1}{T-t}\int_t^T h(s, X_{x,s}(\omega))ds$, $\omega \in \Omega$. Then assume h is convex in x and r is deterministic, $r = r(t)$. Then the pertaining option is increasing in volatility.*

Proof. Using a Feynman-Kac representation, we can write

$$\begin{aligned} F(t,x) &= \frac{1}{T-t}\int_t^T \mathcal{T}[e^{\int_t^T \mathcal{L}ds} h(u,x)]du \\ &= \int_t^T \mathcal{T}[e^{\int_t^u \mathcal{L}ds}]h(u,x)e^{-\int_u^T rds}du. \end{aligned}$$

Thus we see that the mean option is a linear superposition of European contracts, all of them monotone in volatility according to Corollary 3.1. Hence we are done. □

Corollary 4.2. *Assume that we are given an 'Asian' or 'mean' 'put' contract on a traded asset* $g = \frac{1}{T-t}\int_t^T h(s, X_{x,s}(\omega))ds$, $\omega \in \Omega$. *Assume that* $(x\partial_x - 1)he^{-\int_u^T r(s,x)ds} \leq 0$ *or* $(x\partial_x - 1)he^{-\int_u^T r(s,x)ds} \geq 0$. *Then the price of this option is decreasing in rate or increasing in rate respectively for the two different cases.*

Proof. Since we have already proved that a 'mean' option is a superposition of European contracts and we observe that the relevant contract functions in the superposition

$$U(u, x) = h(u, x)e^{-\int_u^T r(s,x)ds}$$

are satisfying $(x\partial_x - 1)U(u, x) \leq 0$ or $(x\partial_x - 1)U(u, x) \geq 0$ we are done by using Theorem 4.1. □

Remark 4.7. *For the general case of Asian options, such as the usual Asian put or call, we do not necessarily expect monotonicity propeties to follow just as easily. Indeed monotonicity in volatility is linked to a spatially bivariate diffusion, which may or may not be convexity preserving. This has to be investigated further and cannot be concluded immediately.*

In as far as American options we can prove the following theorems. For standard properties of American options, optimal stopping problems and pertaining free boundary problems, see the references. We remind the reader of the following definition:

Definition 4.2. *The Frechet differential of a functional* $F(\rho)$ *at* ρ *on a Banach space* X *is defined as the linear functional* DF *satisfying*

$$F(\rho + h) - F(\rho) = DF(h) + o(h)$$

where $\rho \in X$, $h \in X$. *Here* $o(h) = R(h)||h||_X$ *where* R *is a bounded functional in a neighbourhood of the origin and* $||\cdot||_X$ *is the norm on* X. *If this differential exists we call the functional Frechet differentiable.*

Lemma 4.4. *The American option functional is a Frechet differentiable functional w.r.t to the optimal stopping time inside the continuation region up to the boundary.*

Proof. Inside the continuation region up to the boundary we can write

$$F(x, t, \tau) - F(x, t, \tau - h) = E(e^{-\int_{\tau-h}^{\tau} rds}(g(X_\tau)) - E(g(X_{\tau-h}))$$

where we let the contributions at previous times $t < \tau - h$ cancel. Going in the limit $h \to 0$ the expression vanishes, which is equivalent to optimality. Thus we are done proving Frechet differentiability. □

Theorem 4.9 (Main theorem on American Options)**.** *We have that the American option* $F(t, x)$ *satisfying Hypothesis 2.1,2.2 and 2.3 is monotone in* ρ *if and only if*

$$(\frac{\partial \mathcal{L}}{\partial \rho})F \geq 0$$

or

$$(\frac{\partial \mathcal{L}}{\partial \rho})F \leq 0$$

for all t, x *in* $\mathcal{C}_\rho$ *where* $\mathcal{C}_\rho$ *is the continuation region pertaining to parameter* $\rho = \rho(t, x)$. *The two different lines above denote the two different cases respectively, increasing and decreasing. The stopping time* τ *is assumed to be a Frechet diffrentiable functional of* ρ.

PDE proof. We have

$$0 = (\partial_t + \mathcal{L})F(\rho_1)$$
$$0 = (\partial_t + \mathcal{L})F(\rho_2)$$

on $\mathcal{C}_{\rho_1}$ and $\mathcal{C}_{\rho_2}$ respectively. Hence on $\mathcal{C}_{\rho_1} \cap \mathcal{C}_{\rho_2} \neq \emptyset$ we can write by subtracting one equation from the other

$$0 = (\partial_t + \mathcal{L}(\rho_1))F\Delta F + \Delta\mathcal{L}F(\rho_2)$$

where we used the same notation as in the proofs of the previous main theorems for the difference operator Δ. Hence by a standard application of the Feynman-Kac theorem for arbitrary boundaries, see Oksendal, Theorem 9.3.3, we obtain

$$\Delta F = E^{x,t}(e^{\int_t^{\tau_1\wedge\tau_2} r(s,X_s)ds}(F(\rho_1, X_{\tau_1\wedge\tau_2}) - F(\rho_2, X_{\tau_2\wedge\tau_1}))) + E^{x,t}(\int_t^{\tau_1\wedge\tau_2} \Delta\mathcal{L}F(\rho_2)).$$

As $\rho_2 \to \rho_1$ we have $\tau_2 \to \tau_1$, hence

$$\Delta F = E(e^{\int_t^{\tau_1\wedge\tau_2} r(s,X_s)ds} \underbrace{\Delta\rho D_\rho \tau D_\tau F(t,x,\rho_1)|_{\tau_1} + o(\Delta\rho)}_{\to 0}) + E(\int_t^{\tau_1\wedge\tau_2} \Delta\mathcal{L} \underbrace{F(\rho_2))}_{\to F(\rho_1)}.$$

where $D_\tau F(t,x,\rho_1)|_{\tau_1} = D_\tau F(t,x,\tau[\rho_1])|_{\tau_1} = 0$ by optimality. Here we used the chain rule for Frechet differentiable composite mappings, see Choquet-Bruhat et al., page 73.

Thus we ahve that the term that remains after a limiting procedure $\rho_2 \to \rho_1$ is

$$\partial_\rho F = E(\int_t^{\tau_1} e^{\int_t^{\tau_1} r(s,X_s)} \partial_\rho \mathcal{L}F(\rho_1)).$$

since Since we assume

$$\frac{\partial F}{\partial \rho}[\Delta\rho] \geq 0$$

or

$$\frac{\partial F}{\partial \rho}[\Delta\rho] \leq 0$$

respectively

on $\mathcal{C}_{\rho_1} \cap \mathcal{C}_{\rho_1} = \mathcal{C}_{\rho_1}$ this is equivalent to

$$\frac{\partial \mathcal{L}}{\partial \rho}[\Delta\rho]F(\rho) \geq 0$$

or, respectively,

$$\frac{\partial \mathcal{L}}{\partial \rho}[\Delta\rho]F(\rho) \leq 0$$

by the continuous property of the integrand in the expression for $\partial_\rho F$. This concludes the proof.

□

Theorem 4.10. *Assume time and level dependent volatilities and rates. Assume sufficient conditions for the free boundary value problem pertaining to an American option with contract function g to be solvable. Then the American option is increasing in rates if and only if $(x\partial_x - 1)g \geq 0$ for all x in the continuation region and decreasing in rates if and only if $(x\partial - 1)g \leq 0$ for all x in the continuation region.*

Proof. By $(x\partial_x - 1)g \geq 0$ or $(x\partial_x - 1)g \leq 0$ this implies $(x\partial_x - 1)F \geq 0$ or $(x\partial_x - 1)F \leq 0$ by Lemma 4.1. Thus the American option is monotone in rate. Conversely, in the continuation region the option has to satisfy $(x\partial_x - 1)F(x) \geq 0$ or $(x\partial_x - 1)F \leq 0$. Smooth fit proves then the theorem for smooth contracts. By the Hadamard property we are then done for general continuous functions. □

Theorem 4.11. *Assume sufficient conditions for the free boundary problem associated with an American option to be well defined and time and level dependent volatilities and rates. Then the American option is increasing in volatility if and only if it is spatially convex in the continuation region.*

Proof. In view of Main Theorem 4.7 and the fact that $\partial_\sigma \mathcal{L} = \sigma x^2 \partial_x^2$ we are done. □

Remark 4.8. *As E.Ekstrom mentions in his Ph.D. Thesis this was known for deterministic rates previously.*

Remark 4.9. *Theorem 4.9 holds under time and level dependent dividends for which the free boundary value problem is solvable as well. This is seen directly from the proof.*

Theorem 4.12. *Assume time and level dependent volatilities and rates and sufficient conditions for the free boundary value problem pertaining to an American option to be solvable. Assume for this American option that it is a call-like option, r convex, g convex. Then the American option is actually European and monotonically increasing in both rate and volatility. Assume on the other hand that it is a put-like option, r is concave and g is convex. Then the American option is decreasing in rate and increasing in volatility.*

Proof. In view of Theorem 4.6 for European options this is obvious. That it is European in the first case follows from definiteness of the characteristic operator of the diffusion. □

5. Monotonicity of Credit Risky Derivatives

In this section we investigate intensity-based credit risk models. Intensity based models are common in the litterature, see e.g. P.J. Schonbucher[16], Cossin et and Pirotte [15] or Duffie and Singelton[14]. Usually these models are investigated via stochastic methods, something that we in the present paper defer from doing, instead reducing these models to a partial differential equation approach. After this reduction we investigate the monotonicity properties of this derived equation, which is interesting to statistical arbitrage, superhedging and superreplication of claims when either mis-specifying the volatility or rate of the underlying. We specify our model assumptions again since we now explictly allow default of the underlying stock.

5.1. The Black-Scholes Model Revisited.

We assume that the underlying stock or variable follows a one-dimensional stochastic differential equation diffusion under a Kolmogorov measure. Thus we have that

Hypothesis 5.1.

$$dX_t = r(t, X_t)X_t dt + \sigma(t, X_t)X_t dB_t$$

where B_t is Brownian motion, r is the short rate and σ is the volatility of X_t dedined through $\sigma(t,x)^2 X_t^2 dt = Var(dX_t)$. The relevant probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{t\leq s\leq T}, P)$ is as follows: It is the canonical Wiener space where Ω is the space of continuous functions $C([t,T],\mathbb{R})$, $(\mathcal{F}_s^0)$ is the filtration generated by the coordinate process $B_t(\omega)$, $\omega \in \Omega$, P is Wiener measure on $\mathcal{F}_T^0$, $\mathcal{F}$ is the P-completion of $\mathcal{F}_T^0$,and for each s, $\mathcal{F}_s$ is $\mathcal{F}_s^0$ completed with the null sets of $\mathcal{F}$.

Here we have also

Hypothesis 5.2.
- *$r(t,x) \geq 0$ and $\sigma(t,x) \geq \lambda \in \mathbb{R}_+$ are Lipschitz.*
- *The diffusion term $\sigma(t,x)x$ is at most asymptotically linear in growth in the spatial variable:*

$$|\sigma(t,x)| \leq M, M \in \mathbb{R}$$

- *The rate $r(t,x)$ is bounded.*

The usual Black-Scholes equation, without default risk, is then

$$(\partial_t + \frac{\sigma(t,x)^2 x^2}{2}\partial_x^2 + r(t,x)x\partial_x)F(t,x) = rF(t,x)$$
$$F(T,x) = g(x).$$

on the domain $t \in [0,T], x \in \mathbb{R}_+$, see Duffie(1996), section F, Chapter 5. We also assume the following hypothesis:

Hypothesis 5.3. *Here g(x) is at most of polynomial growth and is continuous. The solution is required to be $C^{2,1}$ on the interior of the domain.*

5.2. **Main Results on Credit Risk Superhedging.** We first begin by deriving a parabolic partial differential equation of second order for the price of a credit risk vulnerable option with no recovery.

Theorem 5.1 (PDE for credit risk vulnerable options)**.** *The price F of a credit risk vulnerable option is given by*

$$(\partial_t + \frac{\sigma(t,x)^2x^2}{2}\partial_x^2 + r(t,x)x\partial_x)F(t,x) = (r(t,x)+f(t,x))F(t,x)$$
$$F(T,x) = g(x).$$

on the domain $t \in [0,T], x \in \mathbb{R}_+$. Here f is the default intensity of the obligor under the martingale measure. Time is denoted by t and the value of the stock by x. If Hypothesis 6.2 and 6.3 are satisfied the solution to this equation exists and is unique.

Proof. If we use martingale methods to price the option the option value is given by

$$F(t,x) = E^{(t,x)}(e^{-\int(r+f_O)dt}g(X_T))$$

We note that this directly yields a PDE via Dynkin's formula. That the solution to the equation in the theorem above exists and is unique under hypothesis 6.2 and 6.3 is well known from the PDE litterature. See the appendix below. □

Remark 5.1. *Incidentally, another equation can be derived in another interesting formalism. Say we set two different driving Brownian motions B_1, B_2 and model the default intensity as a Markov process;*

$$dX = r(t,X_t,f_t,Z_t)X_tdt + \sigma(t,X_t,f_t,Z_t)X_tdB_1,t$$
$$df_t = \mu_f(t,X_t,f_t,Z_t)dt + \sigma_f(t,X_t,f_t,Z_t)dB_{2,t}$$
$$dZ_t = f_tdt.$$

Then the PDE for the price of a credit risky option is

$$(\partial_t + \sum_{i,j}\frac{C_{ij}}{2}\partial_i\partial_j + rx\partial_x + \mu_f\partial_f + f\partial_Z)F(t,x,f,Z) = r(t,x)F(t,x,f,Z)$$
$$F(T,x,f,Z) = e^{-Z_T}g(x).$$

on the domain $t \in [0,T], x \in \mathbb{R}_+,\ Z \in \mathbb{R}_+,\ f \in \mathbb{R}_+$. Here $r = r(t,x,f,Z)$ and $\mu_f = \mu_f(t,x,f,Z)$ and also $C_11 = \sigma^2x^2$, $C_12 = C_21 = \sigma x\sigma_f\rho_{12}$, $C_22 = \sigma_f^2$, and

$$\rho_{12} = \frac{E(B_1B_2)}{\sqrt{E(B_1^2)E(B_2^2)}}.$$

Our first theorem is then

Theorem 5.2. *It is necessary and sufficient for the option to be decreasing in the obligor intensity that the option contract is non-negative.*

Proof. Using Main Theorem II and the appendix we are done for the first statement. □

We thus turn to investigating when the option is increasing. Specifically, we concentrate on the following issue:

Definition 5.1. *A option is called call-like if t if $(x\partial_x - 1)F(t,x) \geq 0$ and put-like if t if $(x\partial_x - 1)F(t,x) \leq 0$. The contract is called call-like if satisfies the inequality $(x\partial_x - 1)g \geq 0$ and oppositely put-like if $(x\partial_x - 1)g \leq 0$. This is consistent with our previous work on European options, for which an option is call-like if and only if it has call-like contract.*

Definition 5.2. *We define*

$$\mathcal{L} = \frac{\sigma^2x^2}{2}\partial_x^2 + r(x\partial_x - 1)$$

We note that the diffusion pertaining to our PDE is generated by $\mathcal{L} - f$.

Theorem 5.3. *A credit-risky option such that f is decreasing in stock space is call-like at all times if the contract is call-like. Also, if f is increasing, then a decreasing put-like contract yields a put-like option.*

Proof. We begin by approximating the coefficients and contract by smooth functions. By the methods in the previous sections combined with S.Janson and J.Tysk in the references we notice that it suffices at time $t-\Delta t$, Δt small, to check if the commutator $[x\partial_x - 1, \mathcal{L} - f] = C$ is positive or negative when acting on the appropriate function $F(t,x)$ modulo the action of some second degree elliptic operator on $(x\partial_x - 1)F(t,x)$. We have

$$CF(t,x) = \mathcal{K}(x\partial_x - 1)F(t,x) - x\partial_x(f)F(t,x)$$

where $\mathcal{K}$ is elliptic. We conclude that if F is call-like at t it will be call-like at $t - \Delta t$ if f is decreasing. On the other hand if f is increasing and the option is put-like at time t then it will be put-like at some time $t - \Delta t$ before that. Using the Hadamard well-posedness theorem in the appendix on Hadamard properties we are now done. □

Lemma 5.1. *A positive option with call-like contract is spatially increasing if the intensity f is spatially decreasing.*

Proof. Since then $x\partial_x F \geq F$ from Theorem 6.3 we directly obtain from $F \geq 0$ the desired inequality. □

Theorem 5.4. *A credit risk vulnerable option is increasing in rate if and only if it is call-like and, assuming zero stock default probability(but of course non-zero obligor default intensity), decreasing in rate if and only if it is put-like. Specifically, if the contract is call-like and the intensity f is decreasing then it is increasing in rate. Also that it has put-like contract, zero stock default probability and that the intensity f is increasing is sufficient for the option to be decreasing in rate. Here we assume Hypothesis 6.1,6.2 and 6.3.*

Proof. By Main Theorem II and the appendix this is imminent in view of the previous theorems on call-like and put-like options. □

Theorem 5.5. *If r is convex, g is call-like and convex, f is concave and decreasing, then F is increasing in volatility. If r is deterministic on the other hand then it suffices that g is convex, f is concave and decreasing for F to be increasing in volatility. Here we assume Hypothesis 6.1,6.2 and 6.3.*

Proof. By Main Theorem II we realise directly that it is necessary and sufficient that the option be spatially convex in order for monotonicity in volatility to hold. We proceed following the previous sections in the references as usual with a commutator argument by first approximating the coefficients of the PDE with smooth coefficients to prove or derive sufficent conditions.

$$\begin{aligned}[x^2\partial_x^2, \mathcal{L} - f]F(t,x) &= \mathcal{K}x^2\partial_x x^2 F(t,x) + (x^2\partial_x^2 r)(x\partial_x - 1)F(t,x) \\ &\quad -[x^2\partial_x^2, f]F(t,x)\end{aligned}$$

Since $-[x^2\partial_x^2, \overline{f}]F(t,x) = -(x^2\partial_x^2 f + 2x^2\partial_x f\partial_x)F(t,x)$ we are done in view of our assumptions for the smooth case, since F must be increasing at all times by Lemma 5.1. Using the Hadamard well-posedness Theorem in the appendix we also have the result for continuous contracts and coefficients of the PDE. □

Example 5.1. *Consider the standard Black-Scholes model with constant default intensity for monomial contract functions $g(x) = x^\alpha$, $\alpha \geq 1$. Then the value of the defaultable contract is given by*

$$F = e^{-f\Delta t}e^{(\frac{\sigma^2}{2}\alpha(\alpha-1)+r(\alpha-1))\Delta t}x^\alpha$$

This has the required properties, it is decreasing in f, increasing(non-decreasing) in r when $\alpha \geq 1$ which exactly corresponds to call-like cases and increasing in the volatiltiy in the same cases. Specifically the vega is

$$V = \sigma\alpha(\alpha-1)\Delta t e^{-f\Delta t}e^{(\frac{\sigma^2}{2}\alpha(\alpha-1)+r(\alpha-1))\Delta t}x^\alpha \geq 0$$

which demonstrates our assertion concerning montonicity in volatility.

Example 5.2. *Let r and σ be positive constants. Consider the CEV models*

$$dX = rXdt + \sigma X^\gamma dB$$

Then as $\Delta t \mapsto 0$

$$\partial_r F \sim \Delta t(x\partial_x - 1)F$$

as is seen by the usual generator of the diffusion. When $\gamma > 1/2$ we have zero stock default probability(but non-zero obligor default probability is assumed), hence the above directly implies that the option is monotone in rate iff it is call-like or put-like at all times. We also have for the case of possible default $\gamma < 1/2$ that the option is increasing in rates if the option is call-like.

Remark 5.2. *We note that monotonicity in volatility requiers a concave default intensity. This tells us that one in practice has to impose a roof for the values of stocks where the default intensity becomes zero and hence restrict the Black-Scholes equation to a subinterval of the real half-line when we are considering general time and level models of credit risk default intensity. Alternatively we cannot require monotonicity in volatility and perhaps only montonicity in rate.*

6. Appendix: Hadamard well-posedness and Frechet Differentiability

The Black-Scholes equation satisfies Hadamard well-posedness(under some assumptions). Here is a simple proof:
Proof.

$$\begin{aligned}|F(t,x)-F_n(t,x)| &= |E^x(e^{-\int_t^T r(s,X_s)}(g(X_T)-g_n(X_T)))| \\ &\leq E^x(e^{-\int_t^T r(s,X_s)}|g(X_T)-g_n(X_T)|) \\ &\leq E^x(e^{-\int_t^T r(s,X_s)}) essup|g(X_T)-g_n(X_T)| \\ &= B(t,T,x)||g(x)-g_n(x)||_\infty\end{aligned}$$

which was what we wanted to prove. That bond prices are finite is trivial since they are bounded by 1 when the rate is nonnegative. If we allow negative rates, which is something that occurs in several IR models, we must also assume finite bond prices as an additional and actually quite realistic assumption. When considering options on non-traded stochastic processes we always assume finite bond prices.

□

There is also another Hadamard property to be proved, namely the one under successive approximation of coefficients in the Black-Scholes equation. By Frechet differentiability of the Black-Scholes functional this is however obvious, since continouity must follow from differentiability. But if we *do not* assume Frechet differentiability, we must have sufficient conditions for Hadamard continuity to hold in order for our theorems to be valid for the case with continuous coefficients. Here is a theorem, Theorem A.12, from the article by Svante Jansson and Johan Tysk(2003) that gives us what we are looking for.

Theorem 6.1 (Hadamard Property). *Suppose $\mathcal{M}^m$, $m = 1, 2, ...$ is a sequence of Black-Scholes differential operators on $[0,T]\times\mathbb{R}_+$ such that:*

- *The Black-Scholes operator is parabolic(i.e the generator is elliptic) everywhere, i.e $\sigma(t,x) > 0$ everywhere.*
- *The drift satisfies $|\mu-\sigma\lambda| < B(1+1/x)$, the rate is bounded by B, and the volatiltiy satisfies $|\sigma^2| \leq B(1+1/x^2)$, $B \in \mathbb{R}$.*
- *The rate r, drift $(\mu-\lambda\sigma)x$ and variance $\sigma^2 x^2$ are HÃ¶lder(α).*
- *$\mathcal{M}^m$ tends to $\mathcal{M}$ in the sense that the coefficient tend to some continuous coefficients pointwise.*
- *The contract function is at most of polynomial growth.*

Then the solutions to the various PDE's converges in the sense of uniform convergence on compact subsets to the solution of $\mathcal{M}F = 0$.

In the section on credit risk monotonicity we use the following version of the above theorem:

Theorem 6.2 (Hadamard Property, Credit Risk Section). *Suppose $\mathcal{M}^m$, $m = 1, 2, ...$ is a sequence of Black-Scholes differential operators on $[0,T]\times\mathbb{R}_+$ such that:*

- *The Black-Scholes operator is parabolic(i.e the generator is elliptic) everywhere, i.e $\sigma(t,x) > 0$ everywhere.*
- *The drift satisfies $|r| < B(1+1/x)$, the function $r+f$ is bounded by B, and the volatiltiy satisfies $|\sigma^2| \leq B(1+1/x^2)$, $B \in \mathbb{R}$.*
- *That $r+f$, drift rx and variance $\sigma^2 x^2$ are HÃ¶lder(α).*
- *$\mathcal{M}^m$ tends to $\mathcal{M}$ in the sense that the coefficient tend to some continuous coefficients pointwise.*
- *The contract function is at most of polynomial growth.*

Then the solutions to the various PDE's converges in the sense of uniform convergence on compact subsets to the solution of $\mathcal{M}F = 0$.

Basically we just need the HÃ¶lder property of the coefficients to hold under as well as our more restrictive Hypothesis 2.1,2.2 and 2.3 for Hadamard continuity to hold. That a bivariate function f is Holder(α) in x means that

$$|f(t,x) - f(t,y)| \leq K_I|x-y|^\alpha$$

where K_I is a constant depending on the spatial interval I.

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